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PAVLINI CHELVCCII
A S. JOSEPHO LVCENSIS SCHOLARVM PIARVM

QVONDAM ELOQVENTIAE PROFESSORIS IN ARCHI-GYMNASIO ROMANO, POSTEA GENERALIS SVI ORDINIS

INSTITVTIONES ANALYTICAE,

EARVMQVE VSVS

IN GEOMETRIA, CVM APPENDICE DE CONSTRVCTIONE PROBLEMATVM SOLIDORVM.

EDITIO POST TERTIAM ROMANAM
PRIMA IN GERMANIA.

BIBLIOTECA RICCARDI
IN MODENA
S. IX F. 3 N. 24



VIENNÆ AUSTRIÆ,
Typis JOANNIS THOMÆ TRATTNER, CÆSAR. REG. APOST. MAJEST. AULÆ TYPOGR. ET BIBLIOP. 1761.

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AD LECTOREM.

MIRABERIS fortaſſe Eloquentiæ profeſſorem de ALGEBRA ſcribere, cum illa amœnitates, elegantiam & dicendi copiam amet, hæc ſicca, auſtera & tota ad ſeveritatem compoſita eſſe videatur. At mirare potius, me his temporibus de Algebra ipſa ſcribere, in quibus tot præclara de illa extant clariſſimorum virorum volumina; quibus quicquam addere arrogantiæ, quicquam demere, temeritatis eſt. Hoc quidem me, ut verum fatear, diu ancipitem habuit, diuque prohibuit, quominus publici juris facerem, quæ meo tantum privato ſtudio ſelegeram. At vero cum animadverterem magna illa Analyticæ artis Autographa, quæ nobis incomparabiles viri Vieta, Carteſius, Walliſius, Newtonus, Reinau, ac Wolfius denique tradiderunt, ægre in omnium manibus poſſe verſari, cum ob exemplorum paucitatem & pretium, tum ob multam ea percipiendi difficultatem, quæ in his ſubtiliora de Mathematicis & Phyſicis continentur; illa mihi potius conſcripta eſſe videbantur viris jam in calculi ſcientia exercitatis, quam tyronibus, qui faciliora, tum etiam pauciora petunt: quamobrem his ſaltem hunc libellum non inutilem fore judicavi. Itaque

id perpetuo præ oculis habens, me non viris do-
ctis aliquid de Analyſi novum promere, ſed ju-
ventutem a primis ejusdem diſciplinæ rudimentis
inſtituendam ſuſcepiſſe, quæ ſcilicet ultra com-
munem Arithmeticam, Euclidis elementa & pau-
ca Archimedis theoremata progreſſa non eſſet: in
id totus incubui, ut quæ minus dilucide ab aliis
eſſent explicata, aut ad tyronum captum haud
ſatis apta viderentur, ea magis explanarem, ma-
gisque ordinate diſponerem. Idcirco etiam mini-
ma, quæ ad id conferrent, conſectatus ſum: quo-
rum defectus ſæpe ſolet ab incepto curſu diſcen-
tes non ſine multo temporis diſpendio retardare.
Neque enim juvenes omnes ſummo ingenio &
acriori mentis vi donatos cogitare debemus. Sunt
complures mediocri intellectu præditi, quibus
tamen ad ejusmodi ſtudia, unde eorum mens mi-
rum in modum perfici ac roborari poſſit, iter om-
nino intercludi non debet. Adde pleroſque aliis
quoque diſciplinis addictos haud poſſe diem inte-
grum in calculo & Mathematicis meditationibus
ponere. Quid ſi illis divinatione opus ſit, ut au-
ctoris mentem aſſequantur, diuque ſubobſcuri, vel
ambigui ſermonis ſenſum ſpeculari cogantur? Quo
quidem nihil eſt ab omni Matheſi magis alienum;
in cujus laudibus Geometræ omnes perſpicuitati
& evidentiæ primas deferunt. Quo circa videant

quam

quam longe a propoſito aberrent, qui his de rebus obſcure ac perplexo loquentes, ſe tamen in juventutis uſum ſcribere profitentur. Pugnare (pace eorum dicam) ſecum ipſi videntur, dum ſcientiam alioqui reconditam obſcuritate verborum & ambagibus infuſcant, quam potiſſimum ſuſceperant illuſtrandam. At vero cum *in ſcientiis addiſcendis*, ut magnus Newtonus (*a*) inquit *exempla magis, quam præcepta proſint*, in iis tradendis brevitate, quam maxima potui, uſus ſum; omiſſis idcirco demonſtrationibus, ſicubi eas minus neceſſarias putaverim. Exempla vero congeſſi plurima, quæ facilitate non minus ac elegantia ceteris antecellerent, ex quibus non pauca clariſſimis Mathematicis, quos paulo ante nominavi, tum aliis quoque accepta refero; quorum nomina, præſertim ſi pulcherrimæ alicujus methodi auctores extiterint, in Scholiis appoſui, ut una ſimul tyrones in hiſtoria Mathematica erudiantur, tum etiam ut debita unicuique laus tribuatur. *Eſt enim benignum*, ut nos Plinius (*b*) admonet, & *plenum ingenui pudoris fateri, per quos profeceris.* Ceterum quicquid ſit de facilitate ac brevitate methodi a nobis ſuſceptæ, illud ſibi ſtudioſi juvenes perſuadeant, neminem triduo vel quatriduo fieri Algebriſtam; neque inter

am-

(*a*) Arithm. Univ. edit. 3. pag. 179.
(*b*) In præfat. hiſt. Natur.

ambulandum & oſcitanter poſſe calculi ſcientiam obtineri. Non emitur profecto tam vili diſciplina illa, quam ſummi ingenii & ſubtilitatis vir Cardanus (*a*) *Artem Magnam*, quam *univerſæ Matheſeos clavim* Ougthredus (*b*), quam denique Wolfius (*c*) *apicem totius humanæ eruditionis* appellat. An ludus tibi videtur paucis lineis poſſe difficillima quæque in omni naturali ſcientia problemata ſolvere, ac Geometrice componere? Enim vero ſedulo inſiſtas oportet in iis maxime accurate calculandis, quæ quatuor, aut quinque prioribus hujus libri capitibus comprehenduntur. Ubi ex his vadis emerſeris, omnia fere tibi plana occurrent, præſertim ſi quempiam viæ ducem habeas: quod ſi neminem (quod ſæpe uſuvenit) at tute ipſe tibi ductor ſis & magiſter, macte animo, μελέτη τὸ πᾶν, inquit Periander. Quem tibi offero libellum diligenter evolve, nec pigeat calamo excipere & ad calculum revocare ſigillatim, quæ inibi traduntur: cito progreſſus in hoc ſtudio non pœnitendos experiere.

(*a*) Oper. Vol. 4. pag. 221.
(*b*) Clavis Mathem. Oxonii an. 1631. & 1693.
(*c*) In præfat. ad Elem. Analyſ.

Errata.

Paginæ.	Lineæ.	Errata.	Legenda.
5.	15.	$\frac{\cdot\cdot}{\cdot\cdot}$	$\begin{matrix}\cdot\cdot\\\cdot\cdot\end{matrix}$
33.	4.	b^3e	b^3c
46.	6.	x	x^m
Ibidem	11.	$a\frac{1}{3}$	$a^{\frac{1}{3}}$
53.	10.	p^m-3	p^{m-3}
Ibidem	16.	$p3-3.$	$p^{3-3}.$
Ibidem	22.	$p^m-3.$	$p^{m-3}.$
57.	21.	$a^{\frac{3}{2}}b^{\frac{3}{2}}$	$a^{\frac{1}{2}}b^{\frac{3}{2}}$
63.	11.	$a^{-1}\times$	$a^{-1}\times$
Ibidem	13.	$\frac{1}{2}a^{-}$	$\frac{1}{2}a^{-1}$
64.	7.	$-3a^2b$	$-3a^2b$
Ibidem	15.	$+6a^2b_2$	$+6a^2b^2$
65.	18.	$a-3$	$a^{-3}.$
Ibidem	19.	$pm-3$	p^{m-3}
Ibidem	20.	$a-5.$	a^{-5}
73.	1.	$a\sqrt{b^3}$	$a\sqrt[6]{b^3}$
124.	3.	$-x+a$ n	$\frac{-x+a}{n}$
150.	10.	$-\frac{803}{27}$	$-\frac{3.08}{27}$
160.	1.	$\frac{18}{27}$	$\frac{81}{27}$

Ibi-

Paginæ.	Lineæ.	Errata.	Legenda.
Ibidem	4.	$\frac{18}{27}$	$\frac{81}{24}$
181.	0.	x_3	x^3
191.	24.	$x==2$	$x=-2$
194.	5.	$-_4$	$-\frac{1}{4}$
196.	5.	$x \times 1 =$	$x \times 1 = x$
206.	7.	per cc	per $\frac{1}{4}$cc
233.	vltima	$+qf^3$	$-qf^3$
272.	vltima	$\sqrt{5}$	$\sqrt{5}$.
275.	13.	$\overline{a-x^2+d}^2$	$\overline{a-x^2+d^2}^2$
Ibidem	15.	$-2d$	$-2d^2$
280.	14.	$+\frac{1}{2}cc$	$+\frac{1}{4}cc$
302.	8.	x	x^2
304.	14.	& $\frac{14}{15}$	& $3\frac{14}{16}$
324.	20.	$\times AE$	$+AE$
325.	17.	AB	AE
326.	13.	$\times \frac{1}{2}a$	$+\frac{1}{2}a$
336.	10.	GP	FG

IN-

INSTITUTIONES
ANALYTICÆ.

DEFINITIONES.

I. ALGEBRA, ſeu ANALYSIS SPECIOSA, quæ a quibuſdam MATHESIS UNIVERSALIS dicitur, eſt methodus reſolvendi problemata circa quantitatem.

II. *Quantum* dicitur omne id, quod augeri & minui poteſt, ut numerus, linea, ſuperficies, corpus, tempus, motus, velocitas &c.

III. Quantitates Analyticæ exprimuntur per Alphabeti literas, ſed hoc diſcrimine: literæ priores a, b, c, d, &c. adhibentur ad exprimendas quantitates *cognitas*, ſeu *datas*; poſtremæ vero x, z, y, v, ad *incognitas* & *quæſitas*.

IV. Conducit etiam memoriæ quantitates tam cognitas, quam incognitas deſignare per primam literam rei, quam ſignificant, ut numerum per n, ſummam per s, tempus per t, velocitatem per v, circuli radium per r, tangentem per t &c.

V. Multiplicia, vel ſubmultiplicia quantitatum exprimi ſolent indeterminatè & in genere per literas *m*, *n*, *r*, ut *mx*, *ny*, *rz*, prout *m*, *n*, *r* numeros integros, vel fractos deſignant. In particulari vero & in ſpecie exprimuntur per numeros integros, vel fractos, ut $2x$, $3z$, $4y$, & ſignificant duplum, triplum, quadruplum quantitatum *x*, *z*, *y*. Quemadmodum $\frac{1}{2}a$, $\frac{1}{3}b$, $\frac{1}{4}c$, vel etiam $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$ ſignificant partem aliquotam quantitatum *a*, *b*, *c*; hoc eſt dimidium *a*, tertiam partem *b*, quartam partem *c*.

VI. Quanta, quibus præmittitur ſignum PLUS +, *affirmativa*, *ac poſitiva* dicuntur. Quæ vero præferunt ſignum MINUS —, *negativa*, ſeu *privativa*. Quæ autem principio poſita ſunt & omni ſigno carent, habentur pro affirmativis & quaſi ſigno + affecta.

VII. Quantitates Analyticæ, quæ ſignis + & — non ſunt connexæ, dicuntur *ſimplices*, *monomiæ*, & *incomplexæ*, ut ab, abc, $\frac{ab}{c}$. Contra vero $a + b$, $ab - bb$, $ac + m - r$, dicuntur *compoſitæ*, *complexæ*, ſeu *polynomiæ*; & ſi duobus terminis conſtant, ut $a + b$, vel $ab - bb$, *binomiæ*; ſi tribus, ut $ac + m - r$, *trinomiæ*, & ſic deinceps, appellantur.

VIII. Nota *Æqualitatis* eſt $=$, ita ut $a = b$, vel $x = 6$ ſignificet *a* & *b* eſſe æquales, item valorem *x* eſſe 6.

IX. Sed $a > b$ indicat *a majorem* eſſe, quam *b*, Contra $a < b$ denotat *a minorem* eſſe, quam *b*.

X. Proportionalitas Geometrica, ſeu quatuor terminorum Geometrice proportionalium ratio exprimitur per quatuor puncta hoc modo $a . b :: c . d$, nempe *a* eſt ad *b*, ut *c* ad *d*. Si ſit continuata, exprimitur per ſignum ∺. Sic ∺ *a*, *b*, *c* indicat *a* eſſe ad *b*, ut idem *b* ad *c*. Item ∺ 2, 4, 8, 16, 32. &c.

XI. Proportionalitas vero Arithmetica exprimitur per tria puncta hoc pacto $a.\ b \because c.\ d$, nimirum eadem eſt differentia inter a & b, quæ inter c & d, ut 6. 8 ∵ 10. 12.

XII. Numeri, qui quantitates analyticas præcedunt ut $2x$, $3y$, & in trinomio $aa + 2b - 4c$ numeri 2 & 4, dicuntur *Coefficientes*. Ubi nullus eſt numerus, ſemper pro coefficiente intelligitur unitas. Sic ab intelligitur $1ab$, & $2a - b$ intelligitur $2a - 1b$.

CAPUT I.

De Calculo Integrorum.

PROPOSITIO I.

Quantitates ſimplices addere.

I. SI quantitates addendæ eiſdem literis ſint expreſſæ, ſufficit numeros præfixos addere, & numerorum ſummam eidem literæ præfigere. Sic ut addam a ad a, ſcribo $2a$; ut addam b ad $2b$, ſcribo $3b$ &c.

II. Si vero literæ ſint diverſæ, additio fit ſigno +. Sic $a + b$ exprimit ſummam quantitatum a & b. Ecce exempla.

a	a	c	$\frac{1}{2}b$
$2a$	b	d	$2c$
a	a	e	$3x$
$4a$	$2a+b$	$c+d+e$	$\frac{1}{2}b+2c+3x$

SCHOLIUM. *Ordo literarum non attenditur. Summa* $2a + b$ *idem valet ac* $b + 2a$ *Sicuti* 10 + 5 *idem valet ac* 5 + 10 *nempe* 15.

PROPOSITIO II.

Quantitates ſimplices ſubtrahere.

I. SI quantitates iisdem literis ſint expreſſæ, ſubtrahuntur numeri præfixi, & reſiduum eidem literæ præponitur. Ut ſubtraham $2a$ ex $5a$, ſcribo pro reſiduo $3a$. Similiter ſublato b ex $2b$, remanet b. Item ſublato x ab x, remanet o, ſeu nihil.

II. Si quantitates diverſis literis expreſſæ fuerint, ſubtractio fit ſigno —. Ut ſubtraham a ex b, ſcribo $b - a$. Similiter ſublatis $2a$ ex $4y$, reſiduum erit $4y - 2a$. Exempla.

$5a$	$5b$	a	$3c$
$3a$	$2b$	b	$4x$
$2a$	$3b$	$a - b$	$3c - 4x$

Schol. *Intelligenti prima Arithmeticæ communis elementa nulla difficultas eſſe poteſt de duabus præcedentibus propoſitionibus. Nam ſi ponatur* $a = 5$, *erit ſumma* $4a = 5 + 5 + 5 + 5$, *hoc eſt* $4a = 20$, *ut in primo exemplo. Si ponatur* $a = 10$, *&* $b = 4$, *erit ſumma* $2a + b = 20 + 4$, *hoc eſt* 24, *ut in ſecundo exemplo. Similiter ſi ponatur* $a = 10$ *& ſubtrahenda ſint* $3a$ *ex* $5a$, *nempe* 30 *ex* 50, *reſiduum erit* $2a = 20$. *Tandem ſi ponatur* $a = 20$ *&* $b = 15$, *erit* $a - b = 20 - 15$, *hoc eſt* 5, *ut ex primo & tertio ſubtractionis exemplo habetur.*

PROPOSITIO III.

Quantitates ſimplices multiplicare.

I. MUltiplicatio fit per ſimplicem literarum conjunctionem, nullo interpoſito ſigno. Sic ab vel ba

(ordo

(ordo literarum non attenditur) ſignificat productum a in b, vel b in a; unde ſi ponatur $a = 5$ & $b = 6$, erit $ab = 30$.

II. Quantitates multiplicandæ a & b dicuntur *factores* & ab *factum*. Si factoribus præfixi ſint numeri, hi inter ſe multiplicentur, ut in communi Arithmetica, eorumque productum præmittatur facto literarum. Ecce exempla:

ab	ac	$2cd$	$\frac{1}{2}afg$
c	b	$3a$	$\frac{1}{3}cm$
abc	abc	$6acd$	$\frac{1}{6}acfgm$

SCHOL. *Multiplicatio aliquando indicatur per Crucem, quam S. Andreæ vulgo vocant. Sic* a × b *indicat factum* a *in* b, *ſeu* ab. *Ceterum cum in omni multiplicatione ſit unitas ad factorem unum, ut alter ad productum, erit productum quarta proportionalis, hoc eſt* 1. a ∺ b. ab, *proinde optime exprimitur per* ab.

PROPOSITIO IV.

Quantitates ſimplices dividere.

I. REgula eſt, ex dividendo dele diviſorem & habebis quotum. Sic ad dividendum ab per a, deleto a ex dividendo, reſtat b quotus.

II. Si adſint coefficientes, hos ſeorſim divides, ut in vulgari Arithmetica. Sic dividendo $6ab$ per $3b$, quotus erit $2a$; & $\frac{4ax}{2x} = 2a$

III. Quod ſi dividendus & diviſor nullam habeant literam communem, tunc diviſio exprimitur per modum fractionis, ut $\frac{ab}{c}$, $\frac{ax}{z}$, $\frac{2a}{3c}$ &c.

Schol. *Diviſio diſſolvit id, quod fuit multiplicatione compoſitum, proinde diviſor per quotientem multiplicatus reſtituit ipſum dividendum. Nam multiplicato quoto* b *per diviſorem* a, *reſtituitur dividendum* ab. *Atque hinc apparet, cur in diviſione deleto diviſore ex quantitate dividenda, habeatur quotus. Nam quotus & diviſor ſunt duo factores, qui producunt* ab, *& eliſo diviſore* a, *remanet alter ex factoribus pro quoto, nempe* b.

LEMMA.

Quantitates complexas ad ſimpliciores expreſſiones reducere.

I. Si quantitates iisdem literis notatæ habent idem ſignum +, aut —, addantur earum coefficientes, & præponatur ſummæ idem ſignum. Sit reducenda ad ſimpliciorem expreſſionem quantitas $x + 3ab + 2x + 2ab$, additis coefficientibus terminorum ſimilium, reducitur ad $3x + 5ab$. Simili ratione quantitas $y + 2xy - 2d + xy - d$, erit $y + 3xy - 3d$.

II. Cum vero quantitates iiſdem literis expreſſæ habent ſigna diverſa, ſubducendus eſt minor coefficiens a majori, & reſiduo præponendum ſignum majoris. Sit quantitas $4ac + bc - 3bc$, ſubducto $+ bc$ ex $3bc$, & præpoſito ſigno — quantitatis majoris, erit per reductionem, $4ac - 2bc$. Pariter $2ax - ax + 4b - 3b$, erit $ax + b$.

PROPOSITIO V.

Quantitates compoſitas addere.

I. Si quantitates eiſdem literis notatæ habent idem ſignum

ſignum +, aut —, adduntur ſimul, illo eodem præfixo ſigno.

$-a + 3b$	$a - x$	$\frac{1}{2}cb + 2b - 3$
$-a + 2b$	$3a - 2x$	$\frac{1}{2}cb + 3b - 4$
$-2a + b5$	$4a - 3x$	$cb + 5b - 7$

II. Quod ſi eiſdem literis notatæ habent diverſa ſigna, minor quantitas a majori ſubtrahitur, & reſiduo apponitur ſignum majoris.

$3a + 5b$	$a + d$	$ab - 2ac - 5$
$2a - 2b$	$a - 4d$	$ab + 6ac + 2$
$5a + 3b$	$2a - 3d$	$2ab + 4ac - 3$

COROLL. *Manifeſtum eſt, ad addendum* 3a + 5b *ad* 2a — 2b, *ut in primo exemplo, ſcribi poſſe unam quantitatem poſt aliam, nempe* 3a + 5b + 2a — 2b, *& deinde reduci* per Lem. *ad ſimpliciorem expreſſionem; habebitur enim eodem modo* 5a + 3b, *& ſic de aliis.*

PROPOSITIO VI.

Quantitates compoſitas ſubtrahere.

Quantitas ſubducenda additur, mutatis ſignis, quantitati, ex qua ſubduci debet. Subducenda ſit ex quantitate $x + b - c$ quantitas $a + y - d + 1$, erit differentia, ſeu reſiduum $x + b - c - a - y + d - 1$. Similiter in exemplis, quæ ſequuntur, intelligantur mutata ſigna in quantitate ſubtrahenda, erit facile habere reſiduum

$3a + 2b$	$2x - b$	$4c - 3x - 2$
$a + 5b$	$x - 3b$	$2c - 2x + 5$
$2a - 3b$	$x + 2b$	$2c - x - 7$

Coroll. *Patet ſubtractionem quantitatum compoſitarum degenerare in additionem. Nam ad ſubtrahendam ex* $3a + 2b$ *quantitatem* $a + 5b$, *mutatis ſignis in* $a + 5b$, *erit* $- a - 5b$. *Habemus ergo* $3a + 2b - a - 5b$, *quæ* per Lem. *ad ſimpliciorem expreſſionem redacta dant reſiduum* $2a - 3b$, *ut in primo exemplo. Habeat enim quis aureos* 100, *debeat autem alteri aureos* $100 + 50$; *mutatis ſignis in ſumma ſubducenda, habebit ille aureos* $+ 100 - 100 - 50$, *hoc eſt* $- 50$ *aureos de ære alieno.*

PROPOSITIO VII.

Quantitates complexas multiplicare.

DUcantur ſinguli unius quantitatis complexæ termini in ſingulos alterius terminos eo modo, quo factum eſt in multiplicatione ſimplicium; quod commode fit incipiendo a ſiniſtra & pergendo ad dexteram.

Pro ſignis hæc regula obſervetur, *eadem ſigna* faciunt $+$, *diverſa* autem $-$.

Multiplicanda ſint inter ſe *A* & *B*, dico factum eſſe *E*, reſultans ex factis *C* & *D* ſimul additis. Nam primus terminus *a* quantitatis *B* multiplicans ſingulos terminos quantitatis *A*, producit *C*; & ſecundus terminus $- x$ quantitatis ejuſdem *B* multiplicans pariter ſingulos terminos quantitatis *A*, producit *D*, ergo ſi facta *C* & *D* addantur *per Prop.* v. obtinebitur factum *E*. Quod &c.

Ex-

EXEMPL. I.

$$\begin{array}{ll} A & a+b-c \\ B & \quad\; a-x \\ \hline C & aa+ab-ac \\ D & \quad -ax-bx+cx \\ \hline E & aa+ab-ac-ax-bx+cx \end{array}$$

EXEMPL. II.

$$\begin{array}{r} 2a-3bc+1 \\ 3a-2b \\ \hline 6aa-9abc+3a \\ -4ab+6bbc-2b \\ \hline 6aa-9abc+3a-4ab+6bbc-2b \end{array}$$

EXEMPL. III.

$$\begin{array}{r} x+ax-2b+3 \\ x-\frac{1}{2}a+2b \\ \hline xx+axx-2bx+3x \\ -\frac{1}{2}ax-\frac{1}{2}aax+ab-\frac{3}{2}a \\ +2bx+2abx-4bb+6b \\ \hline xx+axx+3x-\frac{1}{2}ax-\frac{1}{2}aax+ab \\ -\frac{3}{2}a+2abx-4bb+6b \end{array}$$

Demonſtr. ſignorum regula. Omnis multiplicatio fit per numeros. Nam quantitatis per quantitatem multiplicatio non eſt niſi iterata ejuſdem quantitatis additio; & per multiplicationis productum aliud non exprimitur, niſi quoties ſumpta fuerit aliqua quantitas. Sic productum $6a$, quod oritur ex 3 in $2a$, exprimit bis ter, ſeu ſexies a eſſe ſumptum; proinde ducendo $+2a$ in $+3$, oritur $6a$, de quo nemini dubium eſt.

At vero ducendo + *in* —, *vel* — *in* +, *oritur* —. Nam multiplicare per numerum negativum eſt tollere, ſeu negare poſitivum, ſicuti multiplicare per numerum poſitivum eſt aliquid ponere. Itaque ducendo + $3a$ in — 2, fit — $6a$, quia bis ter, ſeu ſexies negatur quantitas poſitiva a. Quod erat ſecundum.

Denique — *in* — *dat* +. Nam in hoc caſu quantitas negativa per numerum pariter negativum tollitur & aliquid ponitur. Sicuti tollere debitum eſt ipſum ſolvere ponendo aliquid; unde ducendo — $3a$ in — 2, fit + $6a$, quia dum quantitas negativa bis ter negatur, bis ter aliquid ponitur. Ergo &c.

Vel ſic, — *in* — *dat* +. Nam multiplicando quantitatem negativam per aliam negativam v. g. — $3a$ × — $2b$, illa toties ſubtrahitur, quot ſunt in altera unitates. Negatio enim ſubtractionem importat, ſicuti affirmatio additionem. Hæc autem ſubtractio cum ſit ſubtractio quantitatis negativæ, ſignum negativum mutari debet in affirmativum *per Prop.* VI. proinde reſiduum ſubtractionis (ſeu productum multiplicationis) erit affirmativum. Sic — $3a$ in — $2b$ dat + $6ab$. Nam quantitas — $3a$ bis ſubtrahi debet, quot ſcilicet ſunt unitates in altera quantitate negativa — $2b$; & mutato ſigno negativo in affirmativum, habetur + $6ab$ multiplicationis productum.

Schol. I. *Hanc doctrinam ſi quis nunc penitus non percipiat, parum refert, percipiet poſtea. Interim ultra progrediatur.*

Schol.

SCHOL. II. *Multiplicatio compoſitarum quantitatum exprimitur quoque aliquando per ſignum* ×, *ducta ſupra factores compoſitos linea ſic*, $\overline{a+b-c}\times\overline{a-x}$. *Vel factores ipſi includuntur parentheſi, ut inſigni calculi commodo excogitavit Leibnitius, quo Analyſtæ recentiores utuntur nempe* $(a+b-c)\,(a-x)$. *Similiter* $(a+b-c)$ x, *deſignat productum* $ax+bx-cx$.

PROPOSITIO VIII.

Quantitates complexas dividere.

HIc etiam eadem ſigna ponunt +, diverſa −. Neque refert utrum a ſiniſtra, an vero a dextera initium ducas. Operatio eodem modo peragitur ac in vulgari Arithmetica. Exempla rem declarant. Dividendum eſt polynomium *A* per polynomium *B*, dico quotum eſſe C.

EXEMPLUM. I.

$B\ a+b)$	A	$an+bn$	$-ar-br$	$+a+b$	C
		$-an-bn$	$+ar+br$	$-a-b$	$+n$
		——	——	——	$-r$
		0	0	0	$+1$

Divido *an* per *a* & pono ad dexteram in *C* quotum + *n*, qui habetur *per Prop.* IV. Per hunc multiplico diviſorem *a* + *b*, ejuſque productum *an* + *bn* (mu-

tatis ſignis *per Prop.* vi.) ſubduco ex dividendo A & fit o. Deinde eodem modo per a divido $-ar$, & pono in C quotum $-r$ inventum *per Prop.* iv. per quem multiplicato diviſore $a+b$, oritur productum $-ar-br$ ſubducendum (mutatis ſignis) ex dividendo $-ar-br$ & habetur o. Demum dividendo a per a, quotus eſt 1, per quem multiplico diviſorem $a+b$, & productum $a+b$ (mutatis ſignis) ſubtraho ex dividendo, & nihil remanet. Quotus ergo C eſt $n-r+1$, per quem ſi multiplicetur diviſor $a+b$, reſtituitur dividendus A.

Exemplum II.

Dividendum eſt polynomium B per polynomium C, dico quotum eſſe D.

$C\ 3a-1\,)$	$B\ \ 6aa+3ab+13a-b-5$	D
	$-6aa \qquad\quad +2a$	$+2a$
	$0 \qquad\qquad +15a$	$+b$
		$+5$
Reſid.	$E\ +3ab+15a-b-5$	
	$-3ab \qquad\quad +b$	
	$0 \qquad\qquad 0$	
Reſid.	$F\ +15a-5$	
	$-15a+5$	
	0	

Divi-

Dividendo $6aa$ per $3a$ oritur quotus $2a$ *per Prop.* IV. quem pono in D, & per hunc multiplicans diviſorem C $3a - 1$, habeo productum $6aa - 2a$, ſubducendum (mutatis ſignis) ex dividendo; quo facto habetur reſiduum E. Divido igitur $3ab$ per $3a$, quotus eſt b ponendus in D, per quem multiplicato diviſore C, oritur productum $3ab - b$, ſubtrahendum (mutatis ſignis) ex dividendo, ſeu reſiduo E: quo facto, habetur reſiduum $F + 15a - 5$ loco dividendi. Diviſo itaque $15a$ per $3a$, fit quotus 5, per quem multiplicato diviſore C, & producto $15a - 5$ ſubtracto ex reſiduo F, nihil remanet. Eſt ergo quotus $D\,2a + b + 5$, per quem ſi multiplicetur diviſor C, oritur polynomium B.

EXEMPLUM III.

Dividenda ſit quantitas A per quantitatem B, dico quotum eſſe C.

$B\ x - 2)$	A	$xxx + 4x - 17$	C
		$-xxx + 2xx$	$+xx$
			$+2x$
		0	$+8$
Reſid.	D	$+2xx + 4x - 17$	
		$-2xx + 4x$	
Reſid.	E	$0 + 8x - 17$	
		$-8x + 16$	
		$0 - 1$	

Eſt ergo quotus $C\ xx + 2x + 8 - \frac{1}{x-2}$

Coroll. *Si facta diviſione aliquid remanet, illud ponitur poſt quotientem cum ſigno +, vel —, quo afficitur ut in hoc tertio exemplo* $-\frac{1}{x-2}$ *ſupponendo illi diviſorem, ut fit in communi Arithmetica.*

Schol. I. *Liberum eſt ſumere ex diviſore compoſito, quam libuerit, literam ad dividendum; quæ tamen ſemel aſſumpta ſemper adhibetur, nec mutari poteſt. Sic in primo exemplo pro litera* a *ſumi potuiſſet litera* b. *In ſecundo exemplo loco* 3a, *aſſumi poterat in diviſorem quantitas* — 1, *ſemper enim idem quotus obtinetur.*

Schol. II. *Sæpe numeri, qui præcedunt terminos dividendi, aut diviſoris, impediunt, quo minus fieri poſſit diviſio. Tunc diviſio exprimitur per modum fractionis, ut ſi dividenda ſit* $3ab + c$ *per* $2a - c$, *erit quotus* $\frac{3ab+c}{2a-c}$. *Idem fit cum dividendus & diviſor nullam habent literam communem; ut* $\frac{ad - ac}{d - r}$, *item* $\frac{dx - dc}{a + x}$ &c.

Schol. III. *Aliquando etiam diviſio quantitatum compoſitarum fit includendo diviſorem & dividendum parentheſi, & inter illos apponendo duo puncta; ut* (a + b): c *indicat* a + b *diviſum eſſe per* c. *Similiter* (2ax — ab): (a — c) *deſignat polynomium* 2ax — ab *diviſum eſſe per polynomium* a — c, *quod ſummo Leibnitii ingenio pariter debemus.*

PROPOSITIO IX.

Datæ quantitatis diviſores omnes invenire.

I. ESto numerus 150, cujus ſinguli diviſores quæruntur. Divide illum per 2 & quotum 75 pone ſub ipſo numero dividendo, ut in *A*, diviſorem autem 2 in *B*. Deinde quia totus 75 dividi non poteſt per 2 ſine reſiduo, divide illum per 3 & pone quotum 25 in A, diviſorem autem in *B*. Poſtea quotum 25 (cum adæquate dividi nequeat per 3) divide per 5 & quotum pone in *A*, diviſorem in *B*. Demum quotus 5 diviſus per 5, dat quotum 1, ponendum in *A*, poſito diviſore 5 in *B*. Habentur jam omnes diviſores ſimplices dati numeri, hoc eſt 2, 3, 5, 5.

Ut habeantur diviſores compoſiti, multiplica primum diviſorem 2 per ſecundum 3, & pone productum 6 ad dexteram ejuſdem ſecundi diviſoris 3. Deinde duo primi diviſores & productum 6 multiplicentur per tertium diviſorem 5, & producta 10, 15, 30 ponantur ad dexteram ipſius diviſoris 5. Demum multiplicentur per quartum diviſorem 5 omnes numeri jam inventi, & habebis 25, 50, 75, 150, qui juxta ipſum ultimum diviſorem apponuntur, ut infra patet. Ex compoſitis idem numerus bis non ponitur.

A	*B*				
150	2.				
75	3.	6.			
25	5.	10.	15.	30.	
5	5.	25.	50.	75.	150.
1					

II. Quæ-

II. Quærantur omnes diviſores numeri *M* 120. Diſpone, ut in ſuperiori exemplo factum eſt, omnes diviſores ſimplices infra N, & quotos ſub dato numero infra *M*. Deinde duc primum diviſorem in ſecundum & productum pone ad latus ſecundi diviſoris. Poſtea duo primi diviſores & productum modo inventum ducantur in tertium diviſorem, & ſingula producta, unum poſt aliud ponantur ad latus ipſius tertii diviſoris, & ſic deinceps operando inveniuntur omnes dati numeri diviſores, ut infra apparet.

M	*N*
120	2.
60	2. 4.
30	2. 8.
15	3. 6. 12. 24.
5	5. 10. 20. 40. 15. 30. 60. 120.
1	

III Sit quantitas *aabcd*, cujus omnes diviſores quæruntur. Inveniantur primo omnes diviſores ſimplices *a*, *a*, *b*, *c*, *d*, qui ponantur infra *B*, ſinguli autem quoti infra *A*. Deinde ducendo diviſorem primum in ſecundum, habetur productum *aa*, quod ponitur ad latus ſecundi diviſoris *a*. Similiter ducendo tertium diviſorem *b* in omnes quantitates ſupra exiſtentes, habentur producta *ab*, & *aab*, quæ ſcribuntur ad latus ipſius diviſoris *b*, & ſic deinceps proſequendo habentur omnes diviſores quæſiti, ut infra.

A	*B*
aabcd	*a.*
abcd	*a. aa.*
bcd	*b. ab. aab.*
cd	*c. ac. aac. bc. abc. aabc.*
d	*d. ad. aad. bd. abd. aabd. dc.*
1	*acd. aacd. bcd. abcd. aabcd.*

IV. Quæruntur omnes diviſores quantitatis $2abb-4ac$. Dividatur primo per 2, deinde diviſo quoto $abb-2ac$ per a, remanet quotus indiviſibilis $bb-2c$. Sunt ergo diviſores primi 2, a, & $bb-2c$, quibus multiplicatis ordine jam ſupra explicato, habentur omnes diviſores, ſcilicet

$2abb-4ac$	$2.$
$abb-2ac$	$a.\ 2a.$
$bb-2c.$	$bb-2c.\ 2bb-4c.\ abb-2ac.$
1	$2abb-4ac.$

COROLL. *Hinc patet, quotum indiviſibilem, ut in hoc tertio exemplo* bb—2c, *reponi debere inter diviſores. Nam quælibet quantitas ſuimet diviſor eſt. Unitas quoque primus omnium diviſor intelligitur.*

SCHOL. I. *Diviſoribus primis, ut ſupra inventis, ſi ſingulos binos, ternos, quaternos &c. inter ſe ducas, habebis cum Newtono omnes diviſores compoſitos. Sic numeri* 60. *diviſores primi ſunt* per hanc Prop. 2. 2. 3. 5. *Ex binis compoſiti ſunt* 4. 6. 10. 15; *ex ternis* 12. 20. 30; *ex quaternis* 60. *qui omnes conficiunt ſummam diviſorum* 12.

C SCHOL-

SCHOL. II. *Quod ſi inquiratur tantum ſumma diviſorum dati numeri ex gr. quot diviſores habeat numerus* 60, *incluſo quoque diviſore* 1; *inventis diviſoribus primis, ut ſupra, ſub ſingulis eorum ponitur* 2, *ſi ſint inæquales: ſub binis æqualibus ponitur* 3; *ſub ternis æqualibus* 4, *& ſic deinceps, eorumque productum dat ſummam omnium diviſorum. Primo diviſori* 1 *nihil ſubſcribitur, quia* 1 *in reliquas quantitates ducta nihil novi producit.*

Diviſores primi numeri 60 *ſunt* 2. 2. 3. 5
Pone ſub illis numeros, nempe 3 × 2 × 2
Quorum productum 12 *dat ſummam quæſitam. Similiter primi diviſores numeri* 2310 *ſunt* 2. 3. 5. 7. 11.
Sub ſingulis, quia ſunt inæquales, pono 2, 2 × 2 × 2 × 2 × 2 *quorum productum* 32 *dat ſummam omnium diviſorum numeri* 2310. *Ratio eſt, quia ubi plures ſunt quantitates diverſæ, ſi duæ (omiſſa* 1*) inter ſe ducantur, habentur quatuor diviſores, in quibus comprehenditur* 1, *reliquæ vero continuo duplicant ipſam diviſorum ſummam. Sit productum* abc, *cujus diviſores primi ſunt* 1.a.b.c. *Subſcribe ſingulis (omiſſo diviſore primo* 1*) numerum* 2, *eorum productum* 2 × 2 × 2 = 8 *dat numerum omnium diviſorum. Nam ducto* a × b *fiunt quatuor diviſores nempe* 1.a.b.ab, *qui deinde duplicantur per quantitatem* c *in ſingulos ductam, & fiunt* 8, *ſcilicet* c. ac. bc. abc.

Quod ſi productum ſit aabc, *cujus primi diviſores ſunt* 1.a.a.b.c, *diviſorum ſumma, erit* 12, *hoc eſt* 3 × 2 × 2 = 12, *ut in primo exemplo. Nam quantitates ſimiles* a, a *inter ſe ductæ, non quatuor, ſed tres duntaxat diviſores efficiunt, nempe* 1.a.aa, *qui duplicati per* b *fiunt* 6, *& iterum duplicati per* c *fiunt* 12.

CAPUT II.

De Calculo fractorum.

OMnia fere peraguntur ut in communi Arithmetica. Ne qua tamen tyroni difficultas in hoc calculi genere ſuboriatur, quæ præcipua ſunt, & ad præſentem uſum magis faciunt, breviter illuſtrabimus; adjecta in fine appendice de Calculo Decimali.

AXIOMATA.

1. INtegra quantitas in fractionem degenerat, ſi loco denominatoris ponatur unitas, ut $\frac{ab}{1}$, $\frac{a+b}{1}$ &c.

2. Integrum in fractionem dati denominatoris convertitur, ſi multiplicetur per denominatorem datum, & producto ſupponatur ipſemet denominator, ut ſi a reducenda ſit ad fractionem denominatoris b, erit $\frac{ab}{b}$. Item x ſi reducenda ſit ad fractionem dati denominatoris $a+b$, erit $\frac{ax+bx}{a+b}$.

3. Multiplicatis, aut diviſis per eandem quantitatem tam numeratore, quam denominatore fractionis, fractio valorem non mutat. Sic multiplicando $\frac{a}{b}$ per c fit $\frac{ac}{bc}=\frac{a}{b}$. Item dividendo $\frac{bb}{bc}$ per b, fit $\frac{b}{c}$; & $\frac{ax+bx}{a+b}$ diviſa per $a+b$, fit $\frac{x}{1}$.

 4. Ut

4. Ut multiplicetur fractio per ſuum denominatorem, ſatis eſt delere ipſum denominatorem. Sic ad multiplicandum $\frac{ax}{c}$ per c, ſatis eſt ſcribere ax. Similiter $\frac{bc}{a-b}$ multiplicatum per $a-b$ erit bc; & $\frac{a}{2x}$ multiplicatum per $2x$ erit a; nam $\frac{2ax}{2x} = a$ per 3. *Axioma.*

PROPOSITIO I.

Integrum cum fractione ad unam fractionem reducere.

SIt quantitas $a + \frac{bc}{n}$ reducenda ad unam fractionem. Multiplicetur quantitas integra a per denominatorem fractionis n, fiet fractio quæſita $\frac{an+bc}{n}$. Eadem ratione $\frac{aa}{c} - b$, erit $\frac{aa-bc}{c}$. Ratio patet ex 2. *Axiom.*

PROPOSITIO II.

Fractiones ad ſimpliciorem expreſſionem reducere.

I. SIt fractio $\frac{aab}{ac}$ reducenda ad ſimpliciorem. Dividatur tam numerator, quam denominator per diviſorem communem, nempe per a; quoti hinc inde orti dant

fra-

fractionem ſimpliciorem & priori æqualem per 3. *Axioma* ſcilicet $\frac{ab}{c}$. Eadem ratione $\frac{2abc}{8acd}$ erit $\frac{1b}{4d}$.

II. Quod ſi communis diviſor non ita facile in oculos incurrat, ut in quantitatibus valde compoſitis contingere ſolet; tunc inveniantur *per Prop.* IX. *Cap.* I. omnes tam numeratoris, quam denominatoris diviſores, ex quibus ille pro communi diviſore ſeligatur, qui fuerit numeratori & denominatori communis. Sit fractio reducenda $\frac{aac + abc}{aa - bb}$, omnes quidem numeratoris diviſores ſunt a, c, $a+b$; denominatoris autem ſunt $a - b$ & $a+b$. Diviſor ergo utrique communis $a+b$ eſt diviſor quæſitus, per quem dividendo utrumque terminum datæ fractionis $\frac{aac + abc}{aa - bb}$, fit $\frac{ac}{a - b}$. Eadem ratione $\frac{aac - aad}{cd - dd}$ fit $\frac{aa}{d}$, dividendo per $c - d$ diviſorem communem inventum ut ſupra *per Propoſ.* IX. *Cap. I.*

SCHOL. *Alias regulas inveniendi diviſorem communem, utpote difficiliores, afferre, non putamus eſſe hujus loci.*

PROPOSITIO III.

Fractiones ad eandem denominationem reducere.

I. SInt duæ fractiones reducendæ $\frac{a}{b}$ & $\frac{c}{d}$. Ducantur duo termini fractionis primæ in denominatorem alterius,

nempe $\frac{a}{b}$ in d, & duo termini fractionis ſecundæ $\frac{c}{d}$ in b denominatorem primæ, ſeu (quod idem eſt) multiplicentur per crucem, ut numeri fracti; fient $\frac{ad}{bd}$ & $\frac{cb}{bd}$ ejuſdem nominis & æquales prioribus ex 3. *Axiom.*

II. Sint reducendæ plures; ducantur ambo termini cujuſque fractionis in productum, quod ex ceterarum fractionum denominatoribus reſultat, fient fractiones quæſitæ Sint $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ duc terminos primæ $\frac{a}{b}$ in df, deinde terminos ſecundæ $\frac{c}{d}$ in bf, item terminos tertiæ $\frac{e}{f}$ in bd habebis $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, $\frac{ebd}{bdf}$. Ratio ſequitur ex eod. 3. *Axiom.*

Corol. *Quando denominator unius fractionis exacte dividit denominatorem alterius fractionis, tunc illæ fractiones, ad idem nomen ſatis commode reducuntur multiplicando per illum quotum numeratorem & denominatorem fractionis illius, cujus denominator fuit diviſor. Sint reducendæ* $\frac{\mathrm{ab}}{\mathrm{cd}}$ *&* $\frac{\mathrm{ef}}{\mathrm{c}}$ *quia denominator* c *dividit exacte denominatorem* cd, *multiplico per quotum* d *utrumque terminum fractionis* $\frac{\mathrm{ef}}{\mathrm{c}}$, *& fiunt* $\frac{\mathrm{ab}}{\mathrm{cd}}$ *&* $\frac{\mathrm{edf}}{\mathrm{cd}}$ *ejuſdem nominis. Ut ſi reducendæ ſint* $\frac{5}{8}$ *&* $\frac{3}{4}$, *quia* 4 *dividit exacte* 8, *multiplicando per quotum* 2 *terminum utrumque fractionis* $\frac{3}{4}$, *erunt* $\frac{5}{8}$ *&* $\frac{6}{8}$ *ejuſdem nominis, ut patet.*

PRO-

PROPOSITIO IV.

Additio & ſubtractio fractionum.

I. ADdendæ ſunt fractiones $\frac{a}{c}$ & $\frac{b}{c}$, ſumma erit $\frac{a+b}{c}$ Eadem ratione $\frac{ad}{a+b}$ & $\frac{cf}{a+b}$ ſimul additæ conficiunt ſummam $\frac{ad+cf}{a+b}$.

Si fuerint diverſæ denominationis, ut $\frac{a}{b}$ & $\frac{c}{d}$; reducantur ad eandem *per Prop.* III. fient $\frac{ad}{bd}$ & $\frac{cb}{bd}$, earumque ſumma $\frac{ad+cb}{bd}$.

II. Quod ſi addenda ſint integra cum fractis $a+\frac{ab}{c}$ & $b-\frac{ac}{b}$, addi poſſunt integra integris $a+b$, & fractiones fractionibus $\frac{ab}{c}$ & $-\frac{ac}{b}$ (quæ prius ad eandem denominationem reducendæ ſunt *per Prop.* III.) eritque ſumma quæſita $a+b+\frac{abb-acc}{bc}$.

III. Vel poſſunt integra ad fractionem ſibi annexam reduci *per Prop.* I. nempe $\frac{ac+ab}{c}$ & $\frac{bb-ac}{b}$, quæ redacta

ad

ad eandem denominationem *per Prop.* III. erunt $\frac{abc+abb}{bc}$ & $\frac{bbc-acc}{bc}$, earumque ſumma $\frac{abc+abb+bbc-acc}{bc}$.

IV. Si vero ſubtrahenda ſit $\frac{a}{b}$ ex $\frac{c}{b}$, differentia erit $\frac{c-a}{b}$. Si ſint diverſæ denominationis, ſemper ad eandem prius reduci debent. Sic ut $\frac{a}{b}$ ex $\frac{c}{d}$ ſubtrahi poſſit, reducuntur ad eandem denominationem *per Propoſ.* III. eritque differentia quæſita $\frac{bc-ad}{bd}$.

V. Quod ſi ex integra quantitate x ſubduci debeat fractio $\frac{aa-ab}{a+b}$; reducta prius x ad fractionem ejuſdem denominatoris per 2. *Axioma*, habetur $\frac{ax+bx}{a+b}$, ex qua ſubducta $\frac{aa-ab}{a+b}$, erit reſiduum $\frac{ax+bx-aa+ab}{a+b}$. Similiter ſi ſubtrahenda ſit $b+\frac{cc}{b+d}$ ex $a+b$, redacta prima quantitate ad unam fractionem *per Propoſ.* I. & altera ad fractionem ejuſdem nominis cum prima *per Axioma* 2. erit reſiduum $\frac{ab+ad-cc}{b+d}$, hoc eſt $a-\frac{cc}{b+d}$ *per prop.* VIII. *Cap.* I.

SCHO-

SCHOL. *Cum numerator fractionis pluribus conſtat terminis, juvat aliquando fractionem illam in plures dividere. Sic* $\frac{ab-cd+dd}{a-b}$ *dividi poteſt in* $\frac{ab}{a-b}$, $\frac{-cd}{a-b}$, $\frac{+dd}{a-b}$. *Hoc autem præſertim fit, ubi aliqui termini numeratoris ſunt per denominatorem diviſibiles, alii vero non. Sit enim* $\frac{aa-3ab-bb}{a+b}$, *dividatur in* $\frac{aa-bb}{a+b}$ *&* $\frac{-3ab}{a+b}$, *quia vero* $\frac{aa-bb}{a+b}=a-b$ per Prop. VIII. Cap. I. *erit fractio propoſita* $a-b-\frac{3ab}{a+b}$.

PROPOSITIO V.

Fractiones multiplicare.

I. Sint multiplicandæ duæ fractiones $\frac{a}{b}$ & $\frac{c}{d}$, ducantur inter ſe numeratores, & denominatores pariter inter ſe, fiet productum quæſitum $\frac{ac}{bd}$. Eadem ratione fractio $\frac{a-b}{c}$ multiplicata per $\frac{ab}{x}$ producit $\frac{aab-abb}{cx}$.

II. Si multiplicanda ſit fractio $\frac{a}{b}$ per integrum c, ſatis eſt numeratorem in integrum ducere, eritque productum $\frac{ac}{b}$. Nam integro ſupponitur unitas, & illud ad fractionem eſſe redactum, nempe $\frac{c}{1}$ *per* I. *Axiom.*

III. Vel dividatur (ſiquidem exacte fieri poſſit) denominator fracti per integrum, habebitur productum. Sit $\frac{a}{bc}$ multiplicanda per c, divido bc per c, quotus $\frac{a}{b}$ dat productum quæſitum; nam $\frac{ac}{bc} = \frac{a}{b}$, ut patet. Item ſit multiplicanda $\frac{ab - cd}{ac - ad}$ per $c - d$, divido $ac - ad$ per $c - d$, quotus eſt a, & productum erit $\frac{ab - cd}{a}$.

IV. Si ducenda ſit $a + \frac{aa}{b}$ in $b - \frac{cc}{d}$, reducantur integra ad fractiones ſibi annexas *per Propoſ.* 1. nempe $\frac{ab + aa}{b}$ & $\frac{bd - cc}{d}$, eritque earum productum $\frac{\overline{ab + aa} \times \overline{bd - cc}}{bd}$ hoc eſt $\frac{abbd + aabd - abcc - aacc}{bd}$ ſeu $ab + aa - \frac{acc}{d} - \frac{aacc}{bd}$ *per Schol. Prop. præc.*

V. Fieri etiam poteſt multiplicatio non reductis integris in fractos, ducendo $a \times b$ & habetur ab, tum $\frac{aa}{b} \times b$, & oritur $\frac{aab}{b}$ &c.

$$\begin{array}{c} a + \frac{aa}{b} \\ b - \frac{cc}{d} \\ \hline ab + \frac{aab}{b} - \frac{acc}{d} - \frac{aacc}{bd} \\ \hline ab + aa - \frac{acc}{d} - \frac{aacc}{bd} \end{array}$$

, ut ſupra.

Coroll.

COROLL. *Si fractio multiplicetur per ſuum denominatorem, productum eſt ipſius numerator.* *Sic* $\frac{aa}{a+b} \times a+b$ *dat* aa per 4. Axioma.

SCHOL. *Factum ex integro in fractum, ut* $\frac{ax}{b}$, *quod oritur ex* $\frac{a}{b} \times x$; *vel* $\frac{2ac}{3}$, *quod oritur ex* $\frac{2}{3} \times ac$, *exprimi etiam poteſt ſeorſim hoc pacto* $\frac{a}{b}x$ & $\frac{2}{3}ac$, *quod notetur.*

PROPOSITIO VI.

Fractiones dividere.

I. Ut dividatur $\frac{a}{b}$ per $\frac{c}{b}$, eliſo communi denominatore, divide a per c, erit quotus $\frac{a}{c}$. Eadem ratione $\frac{1aab}{2c}$ diviſo per $\frac{2ad}{3c}$, quotus erit $\frac{3ab}{4d}$. Nam in diviſione fractionum omnia fiunt, ut in multiplicatione, inverſo tamen denominatore minutiæ dividentis; proinde dividendo $\frac{a}{b}$ per $\frac{c}{b}$, inverſo diviſore, factaque multiplicatione, quam docuimus in *præc. Prop.*, erit quotus $\frac{ab}{cb} = \frac{a}{c}$, ut in primo exemplo.

II. Si denominatores diverſi fuerint, eſt eadem omnino regula. Sic $\frac{a}{b}$ diviſa per $\frac{c}{d}$ dat quotum $\frac{ad}{bc}$. Pari ratione $\frac{a-b}{c+d}$ diviſa per $\frac{n}{a}$ dat quotum $\frac{aa-ab}{nc+nd}$.

 III. Si

III. Si fractio $\frac{ac}{b}$ dividenda ſit per integrum c, divide (ſi fieri poteſt, ut in hoc exemplo) numeratorem ac per c, quotus quæſitus erit $\frac{a}{b}$. Similiter $\frac{ad-cd}{a-b}$ dividenda ſit per $a-c$, divido $ad-cd$ per ipſum $a-c$, unde habetur d, & quotus quæſitus erit $\frac{d}{a-b}$.

IV. Vel ſi numerator fractionis dividendæ diviſibilis non eſt, multiplicetur denominator ejusdem fractionis per integrum. Sit $\frac{ac}{b}$ dividenda per d, quia ac non eſt diviſibilis per d, multiplico per ipſum d denominatorem, & habetur quotus quæſitus $\frac{ac}{bd}$. Eadem ratione $\frac{ad-cd}{a-b}$ diviſa per $a+b$, dat quotum $\frac{ad-cd}{aa-bb}$, multiplicando ſcilicet $\overline{a-b}\times\overline{a+b}$. Ratio patet, ſi diviſori integro ſuppoſita intelligatur unitas, nempe $\frac{a+b}{1}$ *per 1. Axiom.* quo inverſo, fit multiplicatio, & producitur quotus, ut ſupra.

V. Si dividi oporteat $a+\frac{aa}{b}$ per $b-\frac{cc}{d}$, reducantur integri ad fractos ſibi adhærentes per *Propoſ. 1.* deinde $\frac{ab+aa}{b}$ dividatur per $\frac{bd-cc}{d}$, invertendo diviſorem, & multiplicando modo ſupra explicato, erit quotus $\frac{abd+aad}{bbd-bcc}$.

SCHOL. *Si fractiones dividendæ ſint valde compoſitæ, juvabit ad majorem facilitatem eas, antequam fiat diviſio, ad terminos ſimpliciores reducere* per Prop. 11. hujus.

PROPOSITIO VII.

Polynomium cum fractionibus dividere.

I. Sit polynomium A cum fractionibus ſibi annexis, quod dividendum ſit per polynomium B pariter cum fractionibus, dico quotientem eſſe $Q = \frac{2}{3}x - \frac{1}{4}a$.

$$A \quad \tfrac{1}{3}axx - \tfrac{1}{2}bxx + \tfrac{2}{3}abx - \tfrac{1}{8}aax + \tfrac{3}{16}abx - \tfrac{1}{4}aab$$
$$0 \qquad 0 \qquad 0 \qquad 0 \qquad 0 \qquad 0$$

$$B. \quad \tfrac{1}{2}ax - \tfrac{3}{4}bx + ab$$
$$Q \quad \tfrac{2}{3}\ x - \tfrac{1}{4}a$$

Nam dividendo $\frac{1}{3}axx$ per $\frac{1}{2}ax$, quotus erit $\frac{2}{3}x$ per *Propoſ. præc.* quem pono in Q, & ſtatim deletur $\frac{1}{3}axx$ ſupponendo illi zerum. Per quotum inventum multiplico diviſoris B reſiduum, hoc eſt $-\frac{3}{4}bx + ab$; & primo quidem $-\frac{3}{4}bx \times \frac{2}{3}x = -\frac{6}{12}bxx = -\frac{1}{2}bxx$, quod (mutato ſigno) ſubtrahitur ex $-\frac{1}{2}bxx$, cui proinde ſuppono zerum. Deinde $\frac{2}{3}x \times ab = \frac{2}{3}abx$, quod pariter ſubtrahitur ex $\frac{2}{3}abx$, ſuppoſito zero.

Rurſus per eundem diviſorem $\frac{1}{2}ax$ divido $-\frac{1}{8}aax$, quotus erit $-\frac{1}{4}a$ *per Prop. præc.* quem pono in Q; & ſtatim deleto $-\frac{1}{8}aax$ per zerum illi ſuppoſitum, multiplico per ipſum quotum reſiduum diviſoris B, eritque primo $-\frac{1}{4}a \times -\frac{3}{4}bx = \frac{3}{16}abx$, ſecundo $-\frac{1}{4}a \times ab = -\frac{1}{4}aab$; quæ facta, mutatis ſignis, ſubtrahuntur ex dividendo, ſupponendo illi zerum, & nihil remanet; proinde habetur quotus $Q\ \frac{2}{3}x - \frac{1}{4}a = \frac{8x-3a}{12}$ *per Prop.* III. & IV. hujus.

II. Sit dividendum polynomium C per polynomium D, dico quotum eſſe $Q = \frac{1}{3}x - \frac{b}{c}$

$$C \quad \frac{x^3}{6b} + \frac{axx}{12b} - \frac{xx}{2c} - \frac{ax}{4c} - \frac{1}{3}cx + b$$
$$\quad\ 0 \qquad\ 0 \qquad\ 0 \qquad\ 0 \qquad\ 0 \quad 0$$

$$D \quad \frac{xx}{2b} + \frac{ax}{4b} - c$$

$$Q \quad \frac{1}{3}x - \frac{b}{c}$$

Nam dividendo $\frac{x^3}{6b}$ (hoc eſt $\frac{xxx}{6b}$) per $\frac{xx}{2b}$, quotus erit *per Prop. præc.* $\frac{1}{3}x$, quem pono in Q, & ſtatim deleto $\frac{x^3}{6b}$ (apponendo illi zerum) multiplico reſiduum diviſoris D, nempe

nempe $+\frac{ax}{4b}-c$, eritque $\frac{1}{3}x \times \frac{ax}{4b} = \frac{axx}{12b}$ & $\frac{1}{3}x \times -c = -\frac{1}{3}cx$, quæ facta ſubtraho ex dividendo, ſupponendo zerum.

Rurſus per eundem diviſorem $\frac{xx}{2b}$ divido $-\frac{xx}{2c}$, & oritur quotus $-\frac{b}{c}$, quem pono in Q; & deleto $-\frac{xx}{2c}$ (ſuppoſito illi zero) multiplico per ipſum quotum reſiduum diviſoris $\frac{ax}{4b}-c$, & duo facta $\frac{ax}{4b} \times -\frac{b}{c} = -\frac{ax}{4c}$ & $-c \times -\frac{b}{c} = +b$ ſubtraho ex dividendo, cumque nihil remaneat, patet quotum eſſe $\frac{1}{3}x - \frac{b}{c} = \frac{cx-3b}{3c}$ *per Prop.* III. *&* IV. *hujus.*

SCHOL. *Pro pleniori hujus ſecundi exempli & ſequentis Propoſ. intelligentia ſciant tyrones* x^2, *vel* x^3, x^4 *&c. idem eſſe ac* xx, *vel* xxx, xxxx *&c. hoc eſt quantitatem* x *elevatam ad quadratum, ad cubum, quadrato quadratum &c. prout infra explicabitur Cap.* III. *At compendii cauſa pro* xx, xxx, *ſcribitur* x^2, x^3 *&c.*

PROPOSITIO VIII.

Valorem fractionis, cujus denominator eſt terminus compoſitus, per infinitos terminos deſignare.

Diviſa ſit quantitas $aa-bb+c$ per $a+b$, quotus erit $a-b$, & remanet c, hoc eſt fractio $\frac{c}{a+b}$ *per Propoſ.*

Propoſ. III. *Cap.* I. cujus fractionis valor per infinitos terminos deſignari poteſt hoc pacto.

Dividatur c per a, erit quotus $\frac{c}{a}$ *per Prop.* VI. *bujus*, quem duc in diviſorem $a+b$, & factum $\frac{ac+bc}{a}$ ſeu $c+\frac{bc}{a}$ *per Schol. Propoſ.* IV. *bujus* ſubtrahe (mutatis ſignis) ex dividendo c, reſiduum erit $-\frac{bc}{a}$ *per Prop.* IV. *bujus*.

Deinde hoc reſiduum $-\frac{bc}{a}$ divide per eundem diviſorem a, quotus erit $-\frac{bc}{aa}$ *per Prop.* VI. *bujus*, quem duc in diviſorem $a+b$, & factum $-\frac{abc-bbc}{aa}$, ſeu $-\frac{bc}{a}-\frac{bbc}{aa}$ ſubtrahe (mutatis ſignis) ex dividendi reſiduo $-\frac{bc}{a}$, remanet $+\frac{bbc}{aa}$.

Rurſus hoc reſiduum dividatur per a; & quotus $\frac{bbc}{a^3}$ ducatur in $a+b$, erit factum $\frac{abbc+b^3c}{a^3}$, ſeu $\frac{bbc}{aa}+\frac{b^3c}{a^3}$ ſubtrahendum (mutatis ſignis) ex reſiduo $+\frac{bbc}{aa}$, & relinquitur $-\frac{b^3c}{a^3}$, & ſic deinceps.

Unde

Unde apparet ratio diviſionem continuandi per infinitos terminos, qui ſunt proportionales, & ſeriem Geometricam conſtituunt, nempe

$$\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3e}{a^4}\ \&c.=\frac{c}{a+b}.$$

CoROLL. I. *Si fiat* a=2, b=1, *&* c=1, *ita ut* $\frac{c}{a+b}=\frac{1}{2+1}=\frac{1}{3}$, *& hi valores ſubſtituantur in terminis ſeriei jam inventis, aut fractio* $\frac{1}{2+1}$ *eodem pacto dividatur, quo diviſa fuit fractio* $\frac{c}{a+b}$, *oritur ſeries Geometrica* $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$ *&c. Similiter ſi fiat* a=3, b=1, *&* c=1, *nempe* $\frac{c}{a+b}=\frac{1}{3+1}$, *oritur ſeries* $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\frac{1}{243}$ *&c. Quæ quidem ſeries terminorum Geometrice decreſcentium, ut patet, exprimere valent quotum quam proxime verum datæ fractionis, modo talis fractionis numerator ſit unitas.*

CoROLL. II. *Series fractionum hujuſmodi continuo decreſcentium, quæ ad verum valorem ſemper magis accedunt, dicuntur* convergentes. *Quod ſi termini continuo creſcant, tunc a vero valore continuo recedunt, ac* divergentes *appellantur. Ut ſi ponatur* a=1, b=2, *&* c=1, *hoc eſt* $\frac{c}{a+b}=\frac{1}{1+2}$, *fit ſeries divergens* 1 − 2 + 4 − 8 + 16 *&c. quæ a valore fractionis* $\frac{1}{1+2}=\frac{1}{3}$ *tanto magis recedit, quanto magis continuatur, ut rem conſideranti patet.*

SCHOL. I. *Hanc doctrinam, quam tanquam inutilem Cruzas in ſua Algebra* Cap. XIV. num. 40. *taxavit, plurimi fecerunt Cl. Viri* Leibnitius ([a]), Jac. Bernullus ([b]), Guido Grandus ([c]), Wolfius, ([d]) & alii.

SCHOL. II. *Cum fractiones Decimales in omni fere Matheſi magni ſint uſus, eaſque nobis in approximatione radicum* CAP. XI. *adhibere mens ſit, brevem de illis notitiam in ſequenti Appendice ſubjicimus.*

APPENDIX

De fractionibus Decimalibus.

DEFINITIO I.

FRactiones Decimales ſunt illæ, quarum denominatores in ratione decupla ab unitate incipiente progrediuntur, nempe 1. 10. 100. 1000. 10000 &c.

Supponatur menſura aliqua, ut pes, virga, libra, vel recta linea diviſa in 10. æquales partes, ſingulæ deinde hæ partes in alias 10 partes, atque hæ ſingulæ rurſus in alias decem, & ſic deinceps, quantum quiſque velit: oriuntur ex hac diviſione partes decimæ, centeſimæ, milleſimæ, centeſimæ milleſimæ &c. quæ vocantur etiam *primæ, ſecundæ, tertiæ, quartæ &c.* iiſque diſtinguendis apponuntur virgulæ, integris autem cifra o. Sic fractio decimalis, $\overset{0}{5}.\overset{I}{8}\ \overset{II}{6}\ \overset{III}{4}\ \overset{IV}{2}$ ſignificat quinque integra, octo primas, ſex ſecundas, quatuor tertias, duas quartas. Sed ſatis eſt virgulam ultimam apponere, & integra puncto diſtin-

(*a*) Acta Erudit. Lipſ. an. 1682. & 1683. (*b*) De Ser. infinit. p. m. 267. (*c*) Theor. Hugen. p. 126. (*d*) Elem. Analy. Cap. 1. Prob. 7.

ſtinguere, ut $5.864\overset{IV}{2}$. Quæ quidem fractio idem valet ac $5 + \frac{8}{10} + \frac{6}{100} + \frac{4}{1000} + \frac{2}{10000}$ ſeu $\frac{58642}{10000}$. At comodi gratia decimales ſcribuntur inſtar integrorum; omiſſo denominatore, modo ſupra explicato, nempe $5.864\overset{IV}{2}$.

COROL. I. *Hinc ſequitur, virgulas illas, ſive apices decimales, qui numeris apponuntur, eſſe loco denominatorum; ut in decimali* $5.\overset{I}{8}\,\overset{II}{6}\,\overset{III}{4}$, *apex unus importat denominatorem* 10, *duo apices denominatorem* 100, *tres denominatorem* 1000 *&c. hoc eſt* $\frac{8}{10}$ $\frac{6}{100}$ $\frac{4}{1000}$.

COROL. II. *Hinc facile decimales ad eandem denominationem reducuntur, addendo tot zeros virgulis decimalibus affectos, quot opus fuerit; ut ſi decimalis* $3.\overset{I}{5}$ *reducenda ſit ad ſecundas, ad tertias vel quartas &c. ſcribitur* $3.\overset{II}{50}$, $3.\overset{III}{500}$, $3.\overset{IV}{5000}$. *Valor enim non mutatur, nam* $\overset{I}{5}=\overset{II}{50}$, $\overset{II}{5}=\overset{III}{50}$ *&c. ut julii* 5 *ſunt aſſes* 50.

COROL. III. *Si integro cifræ quotcunque cum virgulis addantur, ejus valor idem manet, ut ſi ad* 3 *addas* $\overset{III}{000}$, *ut ſit* $3.\overset{III}{000}$, *non mutatur integri valor. Significat enim, ut prius, tres unitates, non autem unitates* 3000; *nam* $\frac{3000}{1000} = 3$.

SCHOL. *Ubi nullum præcedit integrum, ut in decimalibus* $\frac{8}{10}$, $\frac{24}{100}$, $\frac{725}{1000}$ *tunc loco integri ponitur zerus, nempe* 0.8, 0.24, 0.725, *quæ æquivalent præcedentibus* $\frac{8}{10}$, $\frac{24}{100}$ *&c.*

DEFINITIO II.

FRactionum decimalium notæ dicuntur eſſe ejuſdem ordinis, ſeu gradus, quarum iidem ſunt denominatores, vel iidem apices, Sic in decimalibus 0.5679, & 0.045 notæ 6 & 4, item 7 & 5 dicuntur ejuſdem gradus, quia utrique reſpondet idem denominator, ſcilicet $\frac{6}{100}$ & $\frac{4}{100}$, item $\frac{7}{1000}$ & $\frac{5}{1000}$. Nam utrique reſpondet idem apex, qui ſtat loco denominatoris *per Corol.* 1.

DEFINITIO III.

PRogreſſio decimalis interrupta dicitur, cum habentur v. g. partes milleſimæ, ſed partes decimæ, aut centeſimæ nullæ ſunt; ut $4.\overset{I}{2}\overset{IV}{5}$, ubi partes centeſimæ, & milleſimæ deſunt, quæ quidem cifris interpoſitis ſupplentur: ſic eadem decimalis, interpoſitis duabus cifris, erit $4.200\overset{IV}{5}$. Similiter $3.\overset{II}{5}\overset{V}{7}$, fiet $3.0500\overset{V}{7}$. Eadem ratione $\frac{5}{1000}$ ſcribitur $0.00\overset{III}{5}$, & $3 + \frac{45}{10000}$ ſcribitur 3.0045. Semper enim valor eſt idem, ut in *Coroll.* 2. *&* 3. fuit explicatum.

PROPOSITIO I.

Decimales addere. & ſubtrahere.

FRactiones decimales addendæ, vel ſubtrahendæ ſic diſponantur, ut notæ decimales ejuſdem gradus ſibi mutuo reſpondeant *per Defin.* 2.; & ſi progreſſio ſit interrupta,

rupta, ut in ſecundo exemplo ſequenti, ſuppleantur loca vacua *per Defin.* 3. deinde additio & ſubtractio fiant, ut in communi Arithmetica additio & ſubtractio integrorum.

ADDITIONIS EXEMPLA.

$$A \quad 3.24\overset{III}{5}$$
$$B \quad 7.3\overset{II}{9}$$
$$C \quad 10.63\overset{III}{5}$$

$$5.\overset{I}{2}\overset{III}{7} = 5.20\overset{III}{7}$$
$$6.\overset{I}{4}\overset{II}{5} = 6.4\overset{II}{5}$$
$$\text{Summa} \quad 11.65\overset{III}{7}$$

COROLL. *Patet ratio ex dictis: nam ſi decimales* A *&* B *fiant ejuſdem denominationis* per Coroll. 2. *erunt* per Coroll. 1. $A = \frac{3245}{1000}$ *&* $B = \frac{7390}{1000}$, *proinde* $A + B = \frac{3245}{1000} + \frac{7390}{1000} = \frac{10635}{1000} = C\ 10.63\overset{III}{5}$ *per Defin.* 1.

SUBTRACTIONIS EXEMPLA.

$$A \quad 4.57\overset{III}{2}$$
$$B \quad 1.2\overset{II}{9}$$
$$C \quad 3.28\overset{III}{2}$$

$$7.\overset{I}{4}\overset{III}{2} = 7.40\overset{III}{2}$$
$$\overset{II}{3}\overset{III}{5} = 0.03\overset{III}{5}$$
$$\text{differentia} \quad 7.36\overset{III}{7}$$

COROLL. *Ratio eſt eadem ac additionis. Nam ſi decimales* A *&* B *ad eandem denominationem reducantur per* Coroll. 2. *erunt per* Coroll. 1. $A = \frac{4572}{1000}$ *&* $B = \frac{1290}{1000}$ *hinc* $A - B = \frac{4572}{1000} - \frac{1290}{1000} = \frac{3282}{1000} = C\ 3.28\overset{III}{2}$ *per Defin.* 1.

SCHOL. *Ut ex integris decimales ſubtrahi poſſint, adduntur integro tot zeri, quot ſunt apices decimalis ſubtrahendæ. Sic ad ſubtrahendum ex* 8 *integris tres centeſimas, hoc eſt* 0.03^{II}, *adduntur ad* 8 *duo zeri, ut fiat* 8.00^{II} *eritque reſiduum* 7.97^{II}, *ut patet.*

PROPOSITIO II.

Decimales multiplicare.

FRactiones decimales multiplicantur ut integra in communi Arithmetica, nulla habita ratione virgularum decimalium. Sed ad diſtinguendas in producto partes decimales ab integris, adduntur ſimul apices utriuſque factoris; ſumma enim dat numerum notarum decimalium, quæ numerari debent in producto, incipiendo a dextera ſiniſtram verſus.

EXEMPLA.

I.	II.	III.
4.05^{II}	0.745^{III}	0.000356^{VI}
3.2^{I}	42^{II}	0.0048^{IV}
810	1490	2848
1215	2980	1424
12.960^{III}	0.31290^{V}	0.0000017088^{X}

COROLL.

COROLL. I. *Operandi modus ex dictis facile demonſtratur. Nam in primo exemplo* $4.0\overset{II}{5} = \frac{405}{100}$ *&* $3.\overset{I}{2} = \frac{32}{10}$ *per* Defin. I. *Ex communi autem Arithmetica* $\frac{32}{10} \times \frac{405}{100} = \frac{12960}{1000}$ $= 12.96\overset{III}{0}$ *per* Defin. I. & Coroll. I., *quod eſt ejuſdem exempli primi productum.*

COROLL. II. *Patet etiam ex primo exemplo tres tantum notas decimales in facto abſcindi, quia ex factoribus habentur apices tres. In reliquis exemplis quia factores plures apices continent, quam productum notas, ideo ad complendum numerum apicum æqualem, tot adduntur ad ſiniſtram zeri, quot deſunt in facto notæ decimales; unus præterea zerus additur cum puncto ad locum integrorum indicandum.*

COROLL. III. *Quod ſi in alterutro factore progreſſio decimalis ſit interrupta, ut ſi multiplicari oporteat* $4\ \overset{IIII}{5}$ *per* $3.\ \overset{II}{2}$, *primo progreſſio interpoſitis zeris fiat integra per* Defin. 3., *hoc eſt fiat* $4\ \overset{IIII}{5} = 40\overset{III}{5}$, *&* $3.\ \overset{II}{2} = 3.\ 0\ \overset{II}{2}$, *deinde multiplicando* $40\overset{III}{5} \times 3.0\overset{II}{2}$ *habetur productum* $1.\ 2\ 2\ 3\ 1\ \overset{V}{0}$.

SCHOL. *Si factorum unus ſit numerus integer ſine ullo ſibi decimali annexo, in producto numerantur tot notæ, quot apices continet alterius factoris ultima nota dextrorſus.*

PRO-

PROPOSITIO III.

Decimales dividere.

Fiat diviſio, ut in integris fieri ſolet; utque notæ decimales in quoto diſtinguantur, ſubtrahe numerum apicum, quos habet diviſor, a numero apicum, quos habet dividendum, reſiduum dabit numerum notarum decimalium, quæ numerari debent in quoto a dextera ſiniſtram verſus. Si quoti figuræ pauciores ſint, addantur cifræ, ut in III. exemplo.

Exempla.

I. 3^{ii}) $0.1356\overset{\text{v}}{3}$ | 4.521. II. $5.2\overset{\text{ii}}{4}$) $18.86\overset{\text{iii}}{4}$ | $3.\overset{\text{i}}{6}$

3144
—00

III. $27.58\overset{\text{iii}}{9}$) $0.35\overset{\text{iii}}{4}$ ..... | $0.0128\overset{\text{v}}{3}$

27589
—78110
55178
229320
220712
—86080
82767
3313 &c

Co-

Coroll. *Operandi ratio clara eſt. Nam in primo exemplo* per Defin. I. *habetur* $\overset{II}{3} = \frac{3}{100}$ & $0.1356\overset{V}{3} = \frac{13563}{100000}$, *proinde dividendo numeratorem per numeratorem & denominatorem per denominatorem ac ſi eſſent numeri integri, habetur nova fractio* $\frac{4521}{1000} = 4.52\overset{III}{1}$ per Defin. I. *nempe quotus in primo exemplo inventus. In tertio exemplo cum quinquies addita ſit cifra* 0, *dividendi apex ſit* VIII, *a quo diviſoris apice* III *ſubtracto, reſiduum* V *dat quoti apicem.*

Schol: *Quod ſi diviſoris, aut dividendi decimalis progreſſio interrupta ſit, ſiat integra* per Defin. 3. *deinde inſtituatur diviſio, ut ſupra dictum eſt.*

PROPOSITIO IV.

Fractionem quamcunque in partes decimales reducere.

Sit data fractio $\frac{3}{5}$ reducenda in partes v. g. milleſimas; Fiat 5. 3 :: 1000. x, erit per regulam proportionum $x = \frac{3000}{5} = 600$, unde apparet $\frac{3}{5} = \frac{600}{1000} = 0.60\overset{III}{0}$ *per Defin.* I.

Similiter fractio $\frac{3}{7}$ reducenda ſit in partes centeſimas milleſimas, hoc eſt in 100000. Operandum ut ſupra, & invenietur $\frac{3}{7} > 0.42857$, ſed < 0.42858, defectu exiſtente minori quam $\frac{1}{100000}$. Eſt enim fractio decimalis approximans, quæ non exprimit rationem niſi prope veram, ut Cl. Wolfius advertit.

Schol. *Hæ: propoſitio maximum habet uſum, tum in diviſionibus, in quibus habetur reſiduum alicujus momenti, tum etiam in extractione radicum. Nam in utroque caſu ex hac propoſitione haberi poteſt fractio decimalis magis magiſque approximans, quæ exprimat rationem prope veram quoti, ſive radicis quæſitæ. Ratio autem operationis per ſe manifeſta eſt.*

PROPOSITIO V.

Decimales particulas ad fractionem datæ denominationis reducere.

QUæritur quot uncias unius pedis Romani conficiant particulæ decimales $\overset{III}{750}$. Quia pes Romanus dividitur in uncias 12, infertur particulas decimales convertendas eſſe in partes duodecimas. Sunt autem per *Defin.* 1. particulæ $\overset{III}{750} = \frac{750}{1000}$. Fiat jam 1000.750 :: 12.x, erit per regulam proportionum $x = \frac{9000}{1000} = 9$. Habentur ergo $\frac{9}{12}$, proinde $\overset{III}{750} = \frac{9}{12} = \frac{3}{4}$. Igitur particulæ datæ conficiunt novem uncias, ſeu $\frac{3}{4}$ unius pedis Romani.

Similiter ſcire volo, quot aſſes, ſeu quot partes ſcuti Romani, quod in aſſes 100 dividitur, contineant decimales particulæ $\overset{IV}{5610}$. Cum *per Defin.* 1. ſint $\overset{IV}{5610} = \frac{5610}{10000}$, ſi fiat 10000. 5610 :: 100. x, erit per regulam proportionum $x = \frac{561000}{10000} = 56 + \frac{1}{10}$. Continent ergo aſſes $56 + \frac{1}{10}$, hoc eſt aſſes 56, & unius quadrantis Romani ſemiſſem. Ratio per ſe patet.

Schol.

SCHOL. Simon Stevinus ([a]) *decimalium auctor ingenioſiſſimus, eas loco fractionum vulgarium adhibendas propoſuit, ſummo quidem calculi commodo, cum decimales tractentur ſine moleſtia, non ſecus ac integri eſſent numeri, ut vidimus. At recentiores Mathematici* Tacquetus ([b]), Præſtetus ([c]), Reyneau ([d]), Wolfius ([e]), & alii *hoc præclarum inventum illuſtrarunt, additis quoque demonſtrationibus, quæ in auctore deſiderabantur.*

CAPUT III.

De Calculo Exponentiali.

DEFINITIONES.

I. SI quantitas ex. gr. a ſeipſam multiplicet, ex dictis *Propoſ.* 3. *Cap.* 1. fit aa, quod etiam exprimitur per a^2, & dicitur *quadratum* ſeu *ſecunda poteſtas.* Cujus radix, ſeu latus eſt ipſa a, quæ prima etiam poteſtas dicitur. Si aa multiplicetur per idem latus a. oritur aaa, ſeu a^3, nempe *cubus*, aut *tertia poteſtas*. Si aaa rurſus per a multiplicetur, oritur $aaaa$, ſeu a^4, nempe *quadrato quadratum*, aut *quarta poteſtas*, & ſic in infinitum. Atque hinc habetur ſeries continue proportionalium a^1, a^2, a^3, a^4, a^5, a^6, a^7, &c.

II. Numerus poteſtati adſcriptus dicitur *Index*, ſeu *Exponens* illius poteſtatis. Exponit enim quo dimenſionis

F 2 gradu

(a) Oeuvres Mathemat. in f. p. m. 200. (b) Arith. pract. l. 2. Cap. 9.
(c) Elemens des Mathemat. Tom. 1. l. 9. (d) Science du Calcul.
(e) Elem. Matheſeos edit. 2. Tom. 1. Cap. 9.

gradu gaudeat talis poteſtas, & indicat locum, quem occupare debet in ſerie proportionalium. Ut index 4 indicat quatuor dimenſiones ipſius a, & quartum illi locum competere in ordine proportionalium illius ſeriei.

III. Literæ m, n, r, s indicant exponentem potentiæ indeterminatæ, ut a^m, a^n, a^r, quæ poſſunt determinari ad quamlibet potentiam, tertiam, quartam, quintam &c.

IV. Series poteſtatum conſtituit progreſſionem geometricam, quæ procedit in ratione unitatis ad radicem. Exponentes vero progreſſionem Arithmeticam naturalem numerorum 1, 2, 3, 4, 5, &c.

V. Poteſtas, cujus exponens eſt numerus fractus, ut $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$ &c. deſignat radicem ejus poteſtatis, quæ indicatur a denominatore fractionis; nempe $a^{\frac{1}{2}}$ radicem ſecundam, $a^{\frac{1}{3}}$ radicem tertiam, ſeu cubicam &c. unde oritur alia ſeries $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{5}}$, $a^{\frac{1}{6}}$, &c. poteſtatum, quæ dicuntur *imperfectæ*; & idem revera ſignificant ac $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$ &c. quæ ſigna dicuntur *Radicalia*.

VI. Poteſtas, cujus exponens eſt cum ſigno negativo, ut a^{-1}, a^{-2}, a^{-3}, a^{-4} &c. ſignificat unitatem diviſam per talem poteſtatem. Sic a^{-1} idem eſt ac $\frac{1}{a}$, a^{-2} idem ac $\frac{1}{aa}$, a^{-3} ſignificat $\frac{1}{aaa}$ &c. Nam ſi fuerit ex. gr. $\frac{aa}{aaaa}$, eliſo aa tam in numeratore, quam in denominatore, habetur 1,

(nam

(nam $\frac{aa}{aa} = 1$) & remanet $\frac{1}{aa}$, quod æquivalet ipſi a^{-2}. Atque hinc oritur ſeries poteſtatum, quæ dicuntur negativæ, ut a^{-1}, a^{-2}, a^{-3}, a^{-4}, a^{-5} &c. huic æqualis $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, $\frac{1}{a^5}$ &c.

VII. Zero, ſeu nihilum dicitur exponens unitatis, ita ut a^0 ſignificet idem ac unitatem. Hinc poſita progreſſione geometrica ab 1 incipiente, nempe 1, 2, 4, 8, 16 &c. erit $a^0 = 1$, $a^1 = 2$, $a^2 = 4$, $a^3 = 8$, $a^4 = 16$ &c.

COROLL. *Calculus poteſtatum, de quo agitur in hoc capite, fit per* Exponentes *ipſarum; proinde* Calculus Exponentialis *nuncupatur: qui ob poteſtates imperfectas* Defin. 5. *explicatas, extractionem quoque radicum complectitur.*

SCHOL. I. *Nota magnum eſſe diſcrimen inter* 2a *&* aa, *ſeu* a^2. *Nam* 2a *ſignificat duplum ejuſdem* a, *ſeu* a+a; *at vero* a^2 *ſignificat ſecundam poteſtatem ipſius* a. *Hinc poſita* a=3, *erit* 2a=6, *at vero* $a^2=9$.

SCHOL. II. *Etſi nullum exiſtat in natura ſolidum, quod pluribus quam tribus dimenſionibus conſtet, in Algebra tamen alia atque alia concipiuntur ſolida, quorum dimenſionum numerus in infinitum extenditur, ut ſuperiora exempla docent. Uſus autem eſt frequentiſſimus in ſeriebus Geometricis & generatim in doctrina Curvarum.*

PROPOSITIO I.

Potentiam quamcunque per aliam ejuſdem radicis multiplicare, aut dividere.

I. ADde ſimul exponentes, habebis factum.

a^2	$a^{\frac{1}{2}}$	a^{-2}	x^m	y^m
a^3	$a^{\frac{1}{3}}$	a	x	y^n
a^5	$a^{\frac{5}{6}}$	a^{-1}	x^{2m}	y^{m+n}

II. Pro diviſione ſubtrahe exponentem potentiæ dividentis ab exponente potentiæ dividendæ, habebis quotum

$$\frac{a^5}{a^2} \qquad \frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} \qquad \frac{a^2}{a^{-3}} \qquad \frac{y^{m+n}}{y^n} \qquad \frac{x^{2n}}{x^r}$$

$$a^3 \qquad a^{\frac{1}{6}} \qquad a^5 \qquad y^m \qquad x^{2n-r}$$

Demonſtratio. $a^2 = aa$, & $a^3 = aaa$ ex *Defin.* 1. ſed factum ex aa in aaa, ſi per extenſum ſcriberetur, eſſet $aaaaa$, ergo $a^2 \times a^3 = a^{2+3} = a^5$. Similiter $a^{-2} = \frac{1}{aa}$ per *Defin.* 6, & $a^1 = \frac{a}{1}$ per *Ax.* 1, *Cap.* 2. Sed $\frac{1}{aa} \times \frac{a}{1} = \frac{a}{aa}$, *per Prop.* 5. *Cap.* 2. & $\frac{a}{aa} = \frac{1}{a} = a^{-1}$. Ergo multiplicando a^{-2} per a, factum erit a^{-1}. Quod &c.

Simi-

Similili ratione a^5 diviſum per $a^2 = \frac{a^5}{a^2}$ hoc eſt, ſi per extenſum fractio ſcribatur, $= \frac{aaaaa}{aa}$; & ablatis hinc inde aa, reſiduum eſt $aaa = a^3$; ergo ut dividatur a^5 per a^2, quotus erit $a^{\overline{5-2}} = a^3$. Pariter dividere a^2 per a^{-3} idem eſt ac dividere $\frac{aa}{1}$ per $\frac{1}{a^3}$; ſed $\frac{aa}{1}$ diviſum per $\frac{1}{a^3} = \frac{aa \times a^3}{1}$ *per Prop.* 6. *Cap.* 2. ergo dividendo a^2 per a^{-3} provenit quotus a^5. Quod &c.

PROPOSITIO II.

Poteſtatem quancunque ad aliam dati exponentis elevare.

EXponens poteſtatis ducatur in exponentem datum, factum erit exponens poteſtatis quæſitæ. Sit elevanda a^2 ad potentiam tertiam; ductis inter ſe exponentibus 2 & 3, habebis poteſtatem quæſitam a^6. Similiter ſit elevanda poteſtas y^2 ad poteſtatem 6; multiplicatis exponentibus 2 & 6, habebis y^{12} poteſtatem ſextam quæſitam. Si elevanda ſit x^n ad m, habebis x^{mn}. Quod ſi x^n elevanda ſit ad ſecundam, tertiam, & quartam &c. poteſtatem, ſcribitur x^{2n}, x^{3n}, x^{4n} &c.

Demonſtratio patet ex *Defin.* 1. Nam perſpicuum eſt $a^2 \times a^2 \times a^2 = a^{2\times 3} = a^6$, ſic etiam $x^{n\times 2} = x^{2n}$, & $x^{n\times 3} = x^{3n}$ &c.

Co-

Coroll. *Eodem modo elevantur ad poteſtatem dati exponentis quantitates, quæ conſtant facto duarum, vel plurium quantitatum. Sic ſi* ab *elevetur ad poteſtatem ſecundam, erit* $a^2 b^2$, *vel* $\overline{ab}^2$. *Poteſtas tertia quantitatis* $a^2 b^1 c^3$ *erit* $a^6 b^3 c^9$. *Nam* $a^{2\times 3} b^{1\times 3} c^{3\times 3} = a^6 b^3 c^9$. *Poteſtas* m *quantitatis* abcd erit $\overline{abcd}^m$.

Schol. *Poteſtates diſſimiles adduntur, ſubtrahuntur, multiplicantur &c. ut ceteræ quantitates. Sint enim duæ poteſtates* a^m *&* b^n, *earum ſumma erit* $a^m + b^n$, *differentia* $a^m - b^n$, *factum* $a^m b^n$, *quotus* $\frac{a^m}{b^n}$.

PROPOSITIO III.

Binomium ad quancunque potentiam elevare.

I. Sit binomium $a+b$ elevandum ad potentiam v. gr. ſextam. Scribatur in primo termino prima radicis pars evecta ad poteſtatem quæſitam, ut hic, ad 6, quæ erit a^6. In ſecundo termino ſcribatur eadem a evecta ad potentiam unitate minorem, & ducta in alteram radicis partem b, quæ erit $a^5 b$. In tertio ſcribatur eadem radix a evecta ad poteſtatem rurſus unitate minorem ducta in quadratum alterius partis radicis, nempe in b^2, quæ erit $a^4 b^2$, & ſic deinceps, minuendo unitate in quolibet termino poteſtatem primæ partis radicis, & contra augendo unitate poteſtatem ſecundæ partis radicis, donec veniatur ad terminum, in quo prima pars radicis unica tantum dimenſione conſtet, qui erit terminus penultimus, & in ultimo

timo reperiatur ſecunda radicis pars evecta ad eandem poteſtatem quæſitam. Erit igitur poteſtas 6^a dati binomii $a+b$.

$$a^6.a^5b^1.a^4b^2.a^3b^3.a^2b^4.a^1b^5.b^6.$$

II. Pro faciliori autem methodo fieri ſolent ex duabus dati binomii literis duæ progreſſiones Geometricæ, quarum altera a potentia quæſita incipiens, deſcendat usque ad unitatem, altera viciſſim ab unitate aſcendat usque ad eandem potentiam, & multiplicatis ordine terminis unius progreſſionis per terminos alterius, habetur dati binomii potentia quæſita. Quæritur potentia ſexta ipſius $a+b$.

Series I.	$a.^6$	$a.^5$	$a.^4$	$a.^3$	$a.^2$	$a.^1$	1.
Series II.	1.	$b.^1$	$b.^2$	$b.^3$	$b.^4$	$b.^5$	b^6

Poteſtas VI. $a^6+a^5b^1+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$.

III. Coefficientes potentiarum, quos Ougtrhedus *uncias* vocat, ſic reperiuntur. Exponens primi termini dat unciam ſecundi termini, ut in ſuperiori exemplo 6 erit uncia ſecundi termini a^5b. Pro uncia tertii termini, duc unciam modo inventam ſecundi termini, nempe 6, in exponentem 5, quem habet prima pars radicis in eodem ſecundo termino, & productum 30 divide per 2, quotus 15 eſt uncia tertii termini. Deinde duc hanc ipſam unciam 15 in exponentem 4, quem habet in tertio termino radicis prima pars, nempe a^4, & productum 60 divide per 3, quotus dat unciam quarti termini, & ſic deinceps. Itaque potentia ſexta binomii $a+b$ cum unciis ſuis erit $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

IV. Alii vero pro ejuſmodi unciis inveniendis duas Geometricas progreſſiones numerorum componunt, ita ut ex terminis ſibi ordine reſpondentibus fiat fractio, & multiplicando numeratores inter ſe & denominatores pariter inter ſe, habentur numeri unciales quæſiti. Ecce exemplum pro eadem potentia ſexta.

Series I.	6.	5.	4.	3.	2.	1.
Series II.	1.	2.	3.	4.	5.	6.

Hinc $\frac{6}{1} = 6$ eſt uncia ſecundi termini.
$\frac{6}{1} \cdot \frac{5}{2} = \frac{30}{2} = 15$. uncia tertii termini.
$\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3} = \frac{120}{6} = 20$. uncia quarti.
$\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} = \frac{30}{2} = 15$. uncia quinti.
$\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{2}{5} = \frac{6}{1} = 6$. uncia ſexti.
$\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{2}{5} \cdot \frac{1}{6} = 1$. uncia ultimi.

Hac methodo repertæ ſunt unciæ potentiarum 1æ. 2æ. 3æ. 4æ. &c. binomii $a+b$, ut ex ſequenti exemplo apparet.

1æ. $a+b$.
2æ. $a^2+2ab+b^2$
3æ. $a^3+3a^2b+3ab^2+b^3$
4æ. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$
5æ. $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
6æ. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$

SCHOL. I. *Si binomium ſit poſitivum, omnia ſigna ſunt affirmativa, ut patet ex ſuperiori exemplo. Si vero ſecunda binomii pars ſit negativa, ut* $a-b$; *termini, in quibus radix* $-b$ *evecta erit ad poteſtatem imparem,* 1, 3, 5 *&c. afficiendi ſunt ſigno* $-$, *reliqui omnes ſigno* $+$. *Sic poteſtas tertia binomii* $a-b$ *in ſuperiori exemplo fieret* $a^3-3a^2b+3ab^2-b^3$ *ob exponentes impares* b^1, b^3.

SCHOL.

SCHOL. II. *Si numerus aliquis ſive integer, ſive fractus præcedat alterutrum, vel utrumque binomii terminum, ex.gr. ſi elevari debeat ad tertiam potentiam* $a+2b$, *elevetur primo binomium ipſum* $a+b$ *ad tertiam potentiam, quod ſecundum methodum ſuperius allatam erit* $a^3+3a^2b+3ab^2+b^3$. *Deinde numerus* 2, *qui afficit* b, *elevetur ad eandem potentiam, ad quam in ſingulis terminis elevata exiſtit quantitas* b, *erit pro ſecundo termino* $=2$, *pro tertio* $=4$, *pro ultimo* $=8$. *Duc poſtea hos numeros, ſeu poteſtates in coefficientes terminorum, in quibus* b *æquali poteſtate gaudet, habebis* $a^3+6a^2b+12ab^2+8b^3$, *pro tertia poteſtate quæſita binomii* $a+2b$. *Similiter ſi elevanda ſit ad quartam poteſtatem* $2ax+b$, *erit* $16a^4x^4+32a^3bx^3+24a^2b^2x^2+8ab^3x+b^4$.

SCHOL. III. *Aliquando juvat binomium ad datam poteſtatem elevare ſcribendo ad ejus dexteram exponentem poteſtatis quæſitæ, ducta deſuper linea. Sic ut elevetur* $a+b$ *ad quadratum, ſcribitur* $\overline{a+b}^2$; *ut ad cubum,* $\overline{a+b}^3$, *&* *in genere ad quamcunque poteſtatem indeterminatam* m, *ſcribitur* $\overline{a+b}^m$.

COROLL. I. *Hinc ſequitur, ut poteſtas hoc modo efformata ad aliam poteſtatem dati exponentis elevetur, ſatis eſſe exponentem unius ducere in exponentem alterius. Elevanda ſit* $\overline{a+b}^2$ *ad tertiam poteſtatem ſcribitur* $\overline{a+b}^{2\times 3}$ $=\overline{a+b}^6$. *Similiter elevanda ſit* $\overline{a+b}^m$ *ad poteſtatem* n, *ſcribitur* $\overline{a+b}^{mn}$.

Coroll. II. *Item ſequitur, ut duæ poteſtates ejusdem quantitatis inter ſe multiplicentur, ſufficere earum exponentes ſimul addere.* Sic $\overline{a+b}^{2} \times \overline{a+b}^{3}$, *erit* $\overline{a+b}^{2+3} = \overline{a+b}^{5}$. *Similiter* $\overline{a-b}^{m} \times \overline{a-b}^{n} = \overline{a-b}^{m+n}$. *Contra vero ut dividantur, ſatis eſt exponentem dividentis ab exponente dividendi ſubtrahere. Sic* $\overline{a-x}^{5-2} = \overline{a-x}^{3}$, *ſignificat potentiam* $\overline{a-x}^{5}$ *diviſam per potentiam* $\overline{a-x}^{2}$. *Item* $\overline{a+b}^{m-n}$ *eſt quotus reſultans ex diviſione* $\overline{a+b}^{m}$ *per* $\overline{a+b}^{n}$.

PROPOSITIO IV.

Canonem generalem pro quolibet binomio, aut polynomio ad poteſtatem quamcunque elevando aſſignare.

Sit binomium $p+q$ elevandum ad quanlibet poteſtatem indeterminatam m. Fiant duæ progreſſiones Geometricæ ſequentes:

I.	p^{m}.	p^{m-1}.	p^{m-2}.	p^{m-3}.	p^{m-4}.	&c.
II.	1.	q^{1}.	q^{2}.	q^{3}.	q^{4}.	&c.

Quibus inter ſe ductis, habetur formula, ſeu Canon generalis $p^{m} + p^{m-1} q + p^{m-2} q^{2} + p^{m-3} q^{3} + p^{m-4} q^{4}$ &c. Ut inver

inveniantur coefficientes, ſeu unciæ, fiant aliæ duæ ſeries Geometricæ, nempe

I. m. $m-1$. $m-2$. $m-3$. $m-4$. &c.

II. 1. 2. 3. 4. 5. &c.

Fiat fractio, & multiplicatis terminis, ut in *Propoſ. præc.* erit $\frac{m}{1}$ uncia ſecundi termini: $\frac{m.\,m-1}{1.\;2}$ uncia tertii termini: $\frac{m.\,m-1.\,m-2}{1:\;2:\;3}$ uncia quarti termini &c. quibus ſuo loco in ſuperiori Canone generali diſpoſitis, habetur formula generalis pro quocunque binomio, aut polynomio ad quanlibet poteſtatem elevando, nempe $p^m + m\,p^{m-1}q + m\times\frac{m-1}{2}p^{m-2}q^2 + m\times\frac{m-1}{2}\times\frac{m-2}{3}p^{m-3}q^3 + m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}p^{m-4}q^4$ &c.

Patet exponentem m continuo decreſcere, proinde ubi fit $=0$, ibi erit ultimus terminus potentiæ quæſitæ. Nam fiat ex. gr. $m=3$, in quarto Canonis termino erit $p^{m-3} = p^{3-3} = p^0 = 1$.

Sit igitur elevandum ad tertiam poteſtatem binomium $2ax + bb$. Supponatur $2ax = p$, $+bb = q$ & $m = 3$. Loco ipſorum p, q, m ſubſtituantur eorum valores, nempe $2ax$, $+bb$ & 3; adhibitaque formula generali, erit $p^m = 8a^3x^3$, $m\,p^{m-1}q = 12a^2b^2x^2$, $m\times\frac{m-1}{2}p^{m-2}q^2 = 6ab^4x$, demum $m\times\frac{m-1}{2}\times\frac{m-2}{3}p^{m-3}q^3 = b^6$. Eſt igitur tertia poteſtas quæſita $8a^3x^3 + 12a^2b^2x^2 + 6ab^4x + b^6$.

Sit ſecundo trinomium $a+b-c$ elevandum ad quartam poteſtatem. Suppono $a=p$, $+b-c=q$, $m=4$ factaque, ut modo docuimus, debita valorum ſubſtitutione, erit quarta poteſtas quæſita $a^4+4a^3b-4a^3c+6a^2b^2-12a^2bc+6a^2c^2+4ab^3-12ab^2c+12abc^2-4ac^3+b^4-4b^3c+6b^2c^2-4bc^3+c^4$.

PROPOSITIO V.

Poteſtates compoſitas multiplicare.

Multiplicatio poteſtatum compoſitarum nullam habet difficultatem, bene perceptis iis, quæ *Propoſ.* VII. *Cap.* I. & *Prop.* I. *hujus* dicta ſunt, ideoque unum, vel alterum exemplum hoc loco afferre ſatis erit.

I. Multiplicanda ſit quantitas A per B, productum erit C, nempe:

$$\begin{array}{llllll} A & a^2 & -2ab & +b^2 & & \\ B & a^2 & +2ab & -b^2 & & \\ \hline & a^4 & -2a^3b & +a^2b^2 & & \\ & & +2a^3b & -4a^2b^2 & +2ab^3 & \\ & & & -a^2b^2 & +2ab^3 & -b^4 \\ \hline C & a^4 & 0 & -4a^2b^2 & +4ab^3 & -b^4 \end{array}$$

II. Multiplicari oporteat quantitas M $x^2+2ax-a^2+3bx-b^2$ per N x^3-2a. Facilitatis gratia diſponentur termini, ut ſequitur, productum erit P.

$$
\begin{array}{ll}
M & x^2 + 2ax - a^2 \\
 & \phantom{x^2 +{}} 3bx - b^2 \\
\hline
N & x^3 - 2a \\
\hline
 & x^5 + 2ax^4 - a^2x^3 \\
 & + 3bx^4 - b^2x^3 \\
 & \qquad\qquad\qquad - 2ax^2 - 4a^2x + 2a^3 \\
 & \qquad\qquad\qquad\qquad\quad - 6abx + 2ab^2 \\
\hline
P & x^5 + 2ax^4 - a^2x^3 - 2ax^2 - 4a^2x + 2a^3 + 2ab^2 \\
 & + 3bx^4 - b^2x^3 \qquad\qquad - 6abx
\end{array}
$$

Ubi videre eſt terminos, in quibus x ad aliquam poteſtatem elevatur, ordinatim unum ſub alio collocari, ita ut dignitates æquales ſibi reſpondeant.

PROPOSITIO VI.

Poteſtates compoſitas dividere.

Neque multum negotii faceſſit poteſtatum compoſitarum diviſio, ſi quæ dicta ſunt *Prop.* VIII. *Cap.* I. & *Prop.* I. *hujus* ante oculos habeantur.

I. Sit dividenda quantitas $A\ 6a^3 - 15a^2b + 9ab^2$ per $B\ 2a^2 - 3ab$. Diviſo primo termino $6a^3$ per $2a^2$, quotus eſt $3a$ *per Propoſ.* I. *hujus*, quem pone in C, & per hunc multiplica totum diviſorem B: productum $6a^3 - 9a^2b$ ſubtrahitur (mutatis ſignis) ex dividendo A, & remanet $-6a^2b + 9ab^2$; quod divide pariter per $2a^2$, quotus eſt $-3b$ *per Prop. cit.*, quem pone in C, & per ipſum multiplica, ut antea, totum diviſorem B, producitur $-6a^2b$

$-6a^2b+9ab^2$ ſubtrahendum (mutatis ſignis) ex reſiduo prædicto, & nihil remanet. Habetur ergo quotus C $3a-3b$.

$$
\begin{array}{lll|l}
B\ 2a^2-3ab) & A & 6a^3-15a^2b+9ab^2 & C \\
 & & -6a^3+9a^2b & +3a \\
\hline
 & & 0 \quad -6a^2b+9ab^2 & -3b \\
 & & \qquad +6a^2b-9ab^2 & \\
\hline
 & & \qquad 0 \quad 0 &
\end{array}
$$

II. Dividere oporteat M per N, quotus eſt P, ut ſequenti exemplo patet.

$$
\begin{array}{lll|l}
N\ yy-16) & M & y^6-8y^4-124y^2-64 & P \\
 & & -y^6+16y^4 & +y^4 \\
\hline
 & & 0 \quad +8y^4-124y^2 & +8y^2 \\
 & & \qquad -8y^4+128y^2 & +4 \\
\hline
 & & \qquad 0 \quad +4y^2-64 & \\
 & & \qquad\qquad -4y^2+64 & \\
\hline
 & & \qquad\qquad 0 \quad 0 &
\end{array}
$$

PROPOSITIO VII.

Ex poteſtatibus radicem extrahere.

I. Ut extrahatur radix ex poteſtate data, dividatur exponens poteſtatis per exponentem radicis quæſitæ, quotus erit quæſita radix. Quæritur radix ſecunda poteſtatis a^3, diviſo exponente 3 per 2, habetur radix quæſi-

quæſita $a^{\frac{3}{2}}$. Similiter radix tertia ejuſdem poteſtatis erit $a^{\frac{3}{3}} = a^1$. Item radix quinta poteſtatis a^7 erit $a^{\frac{7}{5}}$, ſeu $a^{1+\frac{2}{5}}$ & generaliter $\sqrt[n]{a} = a^{\frac{1}{n}}$; eademque ratione $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

II. Si radix ſit extrahenda ex facto plurium quantitatum, ut $a^6 b^2 c$, eodem modo operandum. Nam diviſis exponentibus ejuſdem $a^6 b^2 c$ per exponentem radicis quæſitæ, habetur radix. Erit ergo ipſius $a^6 b^2 c$ radix ſecunda $a^{\frac{6}{2}} b^{\frac{2}{2}} c^{\frac{1}{2}}$, hoc eſt $a^3 b c^{\frac{1}{2}}$, ita ut $\sqrt{a^6 b^2 c} = a^3 b c^{\frac{1}{2}}$. Sic, $\sqrt{ab} = a^{\frac{1}{2}} b^{\frac{1}{2}}$, ſeu $\overline{ab}^{\frac{1}{2}}$.

Ratio deducitur ex ipſa poteſtatum compoſitione. Nam ſicuti una poteſtas ad aliam elevatur multiplicando earum exponentes *per Prop.* 2. *bujus*; ſic ut radix extrahatur, contrario modo agitur, dividendo ſcilicet exponentem potentiæ per exponentem radicis quæſitæ.

SCHOL. I. *Sed juvabit tyronem ſequentes expreſſiones, quæ apud auctores non raro occurrunt; hic adnotare.* $\sqrt{a^2 b} = a\sqrt{b}$, *ſeu* $a^1 b^{\frac{1}{2}}$; *nam* a *eſt radix ipſius* a^2 & $b^{\frac{1}{2}} = \sqrt{b}$. *Pariter* $\sqrt{a^3 b} = a^{\frac{3}{2}} b^{\frac{1}{2}} = a^{1+\frac{1}{2}} b^{\frac{1}{2}} = a^1 a^{\frac{1}{2}} b^{\frac{1}{2}} = a\sqrt{ab}$, *ubi patet quantitatem partim eſſe rationalem, partim irrationalem. Item* $\sqrt{a^3 b^3} = a^{\frac{3}{2}} b^{\frac{3}{2}} = ab\sqrt{ab}$. *Hæc pauca cæteris dignoſcendis facem præferunt.*

Schol. II. *Hanc novam radices exprimendi rationem excogitavit* Newtonus ([a]) *magno calculi commodo. Nam ſic quantitates irrationales ad formam rationalium reducuntur & eodem modo pertractantur, de quo in* Cap. ſequenti.

PROPOSITIO VIII.

Radicem quadratam ex quantitatibus compoſitis extrahere.

MEthodus extrahendi radices ex compoſitis quantitatibus non differt ab ea, qua in vulgari Arithmetica uti ſolemus. Sed claritatis gratia ſit quantitas A, cujus radix quadrata quæritur.

C	a	$A\ a^2+2ab+b^2+2ac+2bc+cc$	B
D	$2a+b$	$-a^2-2ab-b^2-2ac-2bc-cc$	$a+b+c$
E	$2a+2b+c$	$0 \quad 0 \quad 0$	

Primo radix quadrata quantitatis a^2 eſt a, hanc pone dextrorſum in B, & ſiniſtrorſum in C, ejuſque quadratum a^2 ſubtrahe ex a^2, remanet o. Duplica deinde inventam radicem a, quam pone in D, & per $2a$ divide $+2ab$ datæ quantitatis reſiduum, habebis quotum b, quem adde in B priori radicis termino, & ſiniſtrorſum in D, habebis $2a+b$, quo ducto in radicem b, fit $2ab+bb$, mutatiſque ſignis, ſubtrahe ex quantitate A, remanet o.

Rurſus duplicata radice $a+b$, habebis in E $2a+2b$ diviſorem, per quem divide quantitatem $2ac$, quotus erit c, quo

(a) Epiſt. ad Oldenburg. apud Walliſium Tom. III. pag. 622.

c, quo poſito tam in B, quam in E, multiplica per hanc radicem c modo inventam totam quantitatem E, & productum $2ac + 2bc + cc$ ſubtrahe, mutatis ſignis, ex quantitate reſidua ipſius A, nihil remanet. Proinde $\sqrt{aa + 2ab + bb + 2ac + 2bc + cc} = a + b + c$.

Sit rurſus extrahenda radix quadrata ex quantitate M. Operare ut ſupra, invenies radicem $x + b - 1$, ut in N.

$$\begin{array}{r|l|l}
 & \qquad\qquad M & \\
x & x^2 + 2bx - 2x + b^2 - 2b + 1 & N \\
2x + b & -x^2 - 2bx \qquad\quad - b^2 & x + b - 1 \\
2x + 2b - 1 & \;\; 0 \qquad 0 \qquad\qquad 0 & \\
\hline
 & \text{Reſid.} - 2x - 2b + 1 & \\
 & \qquad + 2x + 2b - 1 & \\
 & \qquad\quad 0 \qquad 0 &
\end{array}$$

COROLL. *Patet ex compoſitione poteſtatum, tum etiam ex earum reſolutione, in omni quadrato quantitatis compoſitæ contineri quadrata partium quantitatum peculiarium, ex quibus conſtat radix, ut in primo exemplo* a^2, b^2, c^2; *unde radix eſt* $a + b + c$; *&* *ulterius contineri duplum rectanguli earundem partium, hoc eſt* $2ab$, $2bc$, $2ac$.

SCHOL. I. *Omne quadratum duplicem habet radicem, unam poſitivam, negativam alteram. Nam* aa *tam oritur ex* $a \times a$, *quam* $-a \times -a$. *Proinde in primo exemplo radix* $a + b + c$ *fieri poteſt negativa* $-a - b - c$, *& in ſecundo* $x + b - 1$ *fieri poteſt* $-x - b + 1$, *& tamen idem quadratum habebitur. Quod pro ſolutione problematum ſecundi gradus ſcire juvabit. Ex quantitate autem*

 nega-

negativa radix quadrata extrahi non poteſt. Nam $-aa$ *neque oriri poteſt ex* $a \times a$, *neque ex* $-a \times -a$, *ut patet* ex Propoſ. VII. Cap. I. *idcirco dicitur quadratum* imaginarium, *quod dari revera non poteſt.*

Schol. II. *Si ex data quantitate radix quadrata extrahi nequit, deſignatur ipſa radix præfigendo quantitati propoſitæ ſignum radicale* $\surd$. *Sic ad extrahendam radicem ſecundam ex* $aa+bb$, *ſcribitur* $\sqrt{aa+bb}$; *& hinc oriuntur quantitates radicales, de quibus in ſequenti Cap.*

PROPOSITIO IX.

Ex quantitate compoſita radicem cubicam extrahere.

I. Sit quantitas A, cujus radix cubica inquiritur.

1. Radix cubica extracta ex a^3 eſt a, *per Prop.* VII., quæ ponatur ſeorſim, ut in B, & per ejus quadrati triplum $3aa$ poſitum in C dividatur ſecundus terminus, nempe $-3aab$, oritur quotus $-b$ pro altera radicis parte ponenda in B.

2. Ex $a-b$ fiat cubus, quem ſubtrahe ex quantitate data A; & cum nihil remaneat, erit radix quæſita $a-b$.

	A	$a^3 - 3aab + 3ab^2 - b^3$	B
C. $3aa$).		$-a^3 + 3aab - 3ab^2 + b^3$	$a-b$
		$0 \quad 0 \quad 0 \quad 0$	

II. Sit extrahenda radix cubica ex quantitate F.

1. Radix cubica primi termini $8a^3$, *per Prop.* VII. eſt $2a$, quam pone in G, & per ejus quadrati triplum $12a^2$ divi-

divide ſecundum terminum quantitatis F, nempe $36a^2$, quotus — 3 dat alteram radicis partem ponendam in G.

2. Ex radice $2a - 3$ fiat cubus $8a^3 - 36a^2 + 54a - 27$, quem ſubtrahe ex quantitate data F; & cum nihil remaneat, ſignum eſt radicem cubicam eſſe $2a - 3$.

$$
\begin{array}{r|l|c}
12a^2\;) & F \quad 8a^3 - 36a^2 + 54a - 27 & G \\
 & \quad -8a^3 + 36a^2 - 54a + 27 & 2a - 3 \\
\hline
 & \quad\; 0 \qquad\; 0 \qquad\; 0 \qquad\; 0 &
\end{array}
$$

Si radix tribus, vel pluribus conſtet terminis, quod ex ipſa terminorum inſpectione facile dignoſcitur, tunc præſtat cubicam radicem inquirere per formulas generales, de quibus in ſequenti Propoſ.

Coroll. *Latere non poteſt methodi hujus ratio niſi eos, qui cuborum geneſin prorſus ignorant. Patet enim cubum ex binomio ex gr. efformatum conſtare cubis utriuſque quantitatis, & ulterius triplo quadrati primæ in ſecundam & viciſſim ſecundæ in primam quantitatem ducti, ut in primo exemplo conſtat cubis* a^3, b^3, *tum etiam* $3a^2b$, & $3b^2a$. *Signa autem, quibus cubi illi ſunt affecti, produnt ſigna eorum radicis, ut in ſecundo exemplo cubus* -27 *indicat radicem negativam* -3.

Schol. I. *Quod ſi ex data quantitate radix extrahi non poſſit modo prædicto, ex. gr. ex* $aa - bb$; *indicatur ſigno radicali cubico, nempe* $\sqrt[3]{aa - bb}$, *quod de aliis quibuſcunque radicibus dictum intelligatur.*

Schol. II. *Fractionum radices habentur, ſi ſeparatim ex numeratore, & denominatore eodem modo extrahantur.*

tur. Juvabit tamen tyronem ad expreſſiones ſequentes animum advertere. Radix quadrata quantitatis $\frac{xx+2ax+aa}{bb}$ *erit* $\frac{x+a}{b}$. *Similiter* $\frac{aa}{bc}$ *erit* $\frac{a}{\sqrt{bc}}$. *Radix cubica quantitatis* $\frac{a^3}{b^3c^3}$ *erit* $\frac{a}{bc}$. *Item* $\frac{a}{\sqrt[3]{dm}}$ *eſt pariter cubica quantitatis* $\frac{a^3}{dm}$.

SCHOL. III. *Si fractio, ex qua radix extrahenda eſt, integro adhæreat, tunc primo integrum ad fractionem ejuſdem nominis reducatur* per Prop. 1. Cap. 2. *Deinde extrahatur, ſi fieri poſſit, radix quæſita ex numeratore & denominatore. Sit extrahenda radix quadrata ex quantitate* $6+\frac{1}{4}$, *facta reductione* per Prop. cit. *habetur* $\frac{25}{4}$, *cujus radix eſt* $\frac{5}{2}=2\frac{1}{2}$. *Similiter ſit quantitas* $\frac{c^2-4c}{4}+1$, *facta reductione* per Prop. cit. *oritur* $\frac{c^2-4c+4}{4}$, *cujus radix* $=\frac{c-2}{2}$ per Prop. 8. hujus.

PROPOSITIO X.

Radices quaſcunque extrahendi methodum generalem aſſignare,

CONſideretur quantitas, ex qua radix eruenda eſt, uti quantitas elevanda ad eam poteſtatem, cujus radix

radix inquiritur. Exponens pro radice quadrata *ex Defin.* 5. erit $=\frac{1}{2}$, pro cubica $=\frac{1}{3}$, pro quadrato quadrata $=\frac{1}{4}$. Exemplis nonnullis res fiet clariſſima.

I. Sit extrahenda radix quadrata ex quantitate $a^2+2ab-2ac+b^2-2bc+cc$. Ex formula generali *Prop.* 4. hujus $p^m+mp^{m-1}q+m\times\frac{m-1}{2}p^{m-2}q^2$ &c. ſumantur duo priores termini (nam alii ſunt inutiles, cum radices extrahendæ ſunt rationales) & ponatur $a^2=p$, $+2ab-2ac$ &c. $=q$, & (quia agitur de radice quadrata) erit $m=\frac{1}{2}$, ideoque primus terminus radicis $=p^m=a^{2\times\frac{1}{2}}=a$. Secundus erit $mp^{m-1}q=\frac{1}{2}a^{2\times-\frac{1}{2}}q$, hoc eſt $\frac{1}{2}a^{-1}\times\overline{2ab-2ac}$, nempe $\frac{1}{2}a^{-1}\times 2ab=a^{-1+1}=a^0b=1b$. Similiter $\frac{1}{2}a^{-}\times-2ac=-a^{-1+1}c=-a^0c=-1c$; ideoque radix quæſita eſt $a+b-c$. Neque enim ulterius eſt progrediendum ubi m fiat $=0$, proinde cum hic habeatur a^0b, & $-a^0c$, habentur duo ultimi radicis termini.

Similiter extrahenda ſit radix quadrata ex quantitate

$$9a^2-12ab+6a+4bb-4b+1.$$

Ex eadem formula generali $p^m+mp^{m-1}q=0$, pone $p=9a^2$, $q=-12ab+6a$ &c. $m=\frac{1}{2}$, erit $p^m=9^{\frac{1}{2}}\times a^{2\times\frac{1}{2}}=3a$, nam $9^{\frac{1}{2}}=3$, *per Defin.* 5., & *Propoſ.* 7. Eſt igitur $3a$ primus terminus. Deinde $mp^{m-1}=\frac{1}{2}\times 9^{\frac{1}{2}-1}$

$=\frac{1}{2}$

$=\frac{1}{2}\times 9^{-\frac{1}{2}}=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}\times a^{2m-1}=\frac{1}{6}a^{-1}\times q=\frac{1}{6}a^{-1}\times -12ab=-\frac{12}{6}a^{-1+1}\times b=-2a^{0}b=-2b$, qui erit ſecundus radicis terminus. Pariter $\frac{1}{6}a^{-1}\times 6a=a^{-1+1}=1a^{0}$. Unde habetur tertius terminus radicis $+1$. Eſt ergo radix quæſita $3a-2b+1$.

II. Quæritur radix cubica quantitatis $a^{3}-3a^{2}b+3ab^{2}-b^{3}$. Pone $a^{3}=p$, & $-3a^{3}b+3ab^{2}-b^{3}=q$, erit (ob radicem cubicam) $m=\frac{1}{3}$. Poſitis igitur valoribus p & q in duobus formulæ generalis terminis $p^{m}+mp^{m-1}q$, habebis $p^{m}=a^{3\times\frac{1}{3}}=a$ primum radicis terminum. Deinde $mp^{m-1}q=\frac{1}{3}a^{3\times-\frac{2}{3}}$, hoc eſt $\frac{1}{3}a^{-2}q=\frac{1}{3}a^{-2}\times-3a^{2}b=-a^{-2+2}b=-a^{0}b=1b$, Atque hic ſiſtendum, ut ſupra monuimus. Radix ergo cubica eſt $a-b$.

III. Quæritur radix quadratoquadrata quantitatis $a^{4}-4a^{3}b+6a^{2}b^{2}-4ab^{3}+b^{4}$. Pone $a^{4}=p$, & $-4a^{3}b+6a^{2}b^{2}-4ab^{3}+b^{4}=q$, erit $m=\frac{1}{4}$ ob radicem quadrato quadratam. Poſitis jam valoribus in duobus formulæ generalis terminis, habebis primum terminum radicis $p^{m}=a^{4\times\frac{1}{4}}=a^{1}$. Pro ſecundo erit $mp^{m-1}q=\frac{1}{4}a^{4\times-\frac{3}{4}}$, hoc eſt $\frac{1}{4}a^{-3}q=\frac{1}{4}a^{-3}\times-4a^{3}b$; unde oritur productum $-a^{0}b$, ex quo admonemur haberi jam ſecundum radicis terminum b, nec eſſe ulterius progrediendum. Eſt ergo radix quæſita $a-b$.

Schol.

SCHOL. *Cum valor exponentis* m *nullibi occurrit* $= 0$, *tunc radix datæ quantitatis erit irrationalis, & extractio radicis continuari poterit in infinitum, ſeu per infinitos terminos, id quod* approximatio radicum *dicitur, de quo in Prop. ſequenti.*

PROPOSITIO XI.

Extractiones radicum per infinitos terminos continuare.

I. Extrahenda ſit ex quantitate $a^2 - b^2$ radix, ex. gr. quadrata, per terminos infinitos A, B, C, D &c. Adhibeantur formulæ generalis tot termini, per quot continuatam vis extractionem radicis, nempe $p^m + mp^{m-1}q + m \times \frac{m-1}{2} p^{m-2} q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} p^{m-3} q^3$ &c. pone $p = a^2$, $q = -b^2$, & (ob radicem quadratam) $m = \frac{1}{2}$, erit

$A \ (= p^m = a^{2 \times \frac{1}{2}}) = a^1$.

$B \ (= mp^{m-1} q = \frac{1}{2} a^{2 \times -\frac{1}{2}}$, hoc eſt $\frac{1}{2} a^{-1} q) = -\frac{b^2}{2a}$.

$C \ (= m \times \frac{m-1}{2} p^{m-2} q^2 = \frac{1}{2} \times -\frac{1}{4}$, hoc eſt $-\frac{1}{8} a^{2 \times -\frac{3}{2}}$, hoc eſt $-\frac{1}{8} a^{-3} q^2) = -\frac{b^4}{8a^3}$.

$D \ (= m \times \frac{m-1}{2} \times \frac{m-2}{3} p^{m-3} q^3 = \frac{1}{2} \times -\frac{1}{4} \times -\frac{1}{2}$, hoc eſt $\frac{1}{16} a^{2 \times -\frac{5}{2}}$, hoc eſt $\frac{1}{16} a^{-5} q^3) = -\frac{b^6}{16a^5}$.

Et ſic deinceps in infinitum. Eſt ergo radix quæſita $A+B+C+D$ &c. $=a-\frac{b^2}{2a}-\frac{b^4}{8a^3}-\frac{b^6}{16a^5}-\frac{5b^8}{128a^7}-\frac{7b^{10}}{256a^9}$ &c.

II. At pro hac formula, qua utitur *Cl. Guiſneus* (a), atque alii, adhibere præſtat formulam Newtonianam magis quidem expeditam, & moleſtiæ fractionum minus obnoxiam videlicet, $P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ$ &c. in qua P ſignificat primum terminum quantitatis illius, cujus radix inveſtigatur. Q reliquos terminos diviſos per primum; $\frac{m}{n}$ indicem dimenſionis, ſeu radicis quæſitæ. Termini jam inventi exprimuntur literis A, B, C, D &c. nempe A exprimit primum radicis terminum $=P^{\frac{m}{n}}$; B, ſecundum $=\frac{m}{n}AQ$; C, tertium $=\frac{m-n}{2n}BQ$, & ſic deinceps. Sed ecce exemplum.

Extrahenda ſit radix ſecunda ex quantitate $\sqrt{c^2+x^2}$ ſeu $\overline{c^2+x^2}^{\frac{1}{2}}$ *per Prop. 7. hujus;* dico radicem eſſe $=c+\frac{x^2}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}$ &c. Nam erit in hoc caſu

$P=c^2$,

(a) Application de l'Algeb. a la Geometrie, edit. 2. Paris pag. xxx.

$P=c^2$, $Q=\frac{x^2}{c^2}$, $m=1$, $n=2$, proinde habetur

$$A\ (=P^{\frac{m}{n}}=c^{2\times\frac{1}{2}})=c.$$

$$B\ (=\frac{m}{n}AQ=\tfrac{1}{2}c\times\frac{x^2}{c^2})=\frac{x^2}{2c}.$$

$$C\ (=\frac{m-n}{2n}BQ=-\tfrac{1}{4}\times\frac{x^2}{2c}\times\frac{x^2}{c^2})=-\frac{x^4}{8c^3}.$$

$$D\ (=\frac{m-2n}{3n}CQ=-\tfrac{1}{2}\times-\frac{x^4}{8c^3}\times\frac{x^2}{c^2})=\frac{x^6}{16c^5}.$$

$$E\ (=\frac{m-3n}{4n}DQ=-\tfrac{5}{8}\times\frac{x^6}{16c^5}\times\frac{x^2}{c^2})=-\frac{5x^8}{128c^7}.$$

Rádix ergo quæſita $=A+B+C+D$ &c. $=c+\frac{x^2}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}$ &c.

Similiter extrahenda ſit radix tertia ex quantitate $\sqrt{\overline{a^3-b^3}}$, ſeu $\overline{a^3-b^3}^{\frac{1}{3}}$ per *Prop. cit.* Procedendo eadem methodo, erit $P=a^3$, $Q=-\frac{b^3}{a^3}$, $m=1$, $n=3$, proinde

$$A\ (=P^{\frac{m}{n}}=a^{3\times\frac{1}{3}}=)\ a^1.$$

$$B\ (=\frac{m}{n}AQ)=-\frac{b^3}{3a^2}.$$

$$C\ (=\frac{m-n}{2n}BQ)=-\frac{b^6}{9a^5}.$$

$$D\left(=\frac{m-2n}{3n}CQ\right)=-\frac{5b^9}{81a^8}.$$

$$E\left(=\frac{m-3n}{4n}DQ\right)=-\frac{10b^{12}}{243a^{11}}$$

Eſt ergo radix quæſita $=A+B+C+D+E$ &c.

$$=a-\frac{b^3}{3a^2}-\frac{b^6}{9a^5}-\frac{5b^9}{81a^8}-\frac{10b^{12}}{243a^{11}} \text{ \&c.}$$

Schol. I. *Hac formula uti quoque poſſumus in diviſione fractionis per terminos infinitos. Sit dividenda* $\frac{1}{a+b}$ (*hoc eſt* $\overline{a+b}^{-1}$ per Def. 6.) *Pone* $P=a$, $Q=\frac{b}{a}$, $m=-1$, $n=1$, *erit* $A\left(=P^{\frac{m}{n}}=a^{-1}\right)=a^{-1}=\frac{1}{a}$.

$$B\left(=\frac{m}{n}AQ\right)=-1\times\frac{1}{a}\times\frac{b}{a}=-\frac{b}{a^2}.$$

$$C\left(=\frac{m-n}{2n}BQ\right)=-1\times-\frac{b}{a^2}\times\frac{b}{a}=\frac{b^2}{a^3} \text{ \&c.}$$

$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}$ &c. *ut in* Prop. 8. Cap. 2.

Schol. II. *Patet jam ex duplici fonte oriri ſeries infinitas terminorum proportionalium, ſcilicet ex diviſione, & extractione radicum. Primo modo quæſitæ fuerunt a* Nicolao Mercatore Holſato (a); *ſecundo a* Newtono (b). *Earum uſus eſt ad obtinendam radicum approximationem, ut vidimus;*

(a) Logarithmo-tecnia Londini 1668. Acta Erudit. Lipſ. an. 1693.
(b) Epiſt. ad D. Oldenburgium ann. 1676. in Tom. 3. Operum Walliſii Analy. per quantit. ſeries &c. in Tom. Philoſophiæ Natur. Amſtelod. ann. 1723.

mus ; ad inveniendas areas & longitudines Curvarum, ad Solidorum ſuperficies dimetiendas, aliaque magni momenti circa curvas Mechanicas expedienda. Nos aliquem ipſarum uſum in reſolutione æquationum experiemur.

CAPUT IV.

De Calculo Radicali.

DEFINITIONES.

I. Quantitas, ſive numerica, ſive literalis, ex qua radix extrahi non poteſt, dicitur potentia imperfecta, quantitas ſurda, & irrationalis ; & a ſigno $\surd$, quod *radicale* dicitur, *radicalis* denominatur, ut $\sqrt[2]{5}$, $\sqrt[2]{a}$, $\sqrt[3]{a-b}$, $\sqrt[4]{ax+bb}$; ſive, ut aliis placet ſcribere, $\sqrt[3]{}(a-b)$, $\sqrt[4]{}(ax+bb.)$

II. Signa radicalia præfixos habent exponentes ejus potentiæ, cujus radicem deſignant. Ubi nullus eſt exponens, pro radice ſecundæ potentiæ accipitur, ut $\surd a$ indicat radicem ſecundam quantitatis a. At $\sqrt[3]{a-b}$ ſignificat radicem tertiam, ſeu cubicam quantitatis $a-b$.

III. Si radicale ſignum præcedat numerus, aut quantitas literalis, ut $2\surd 3$, $a\surd b$ &c. quantitas, quæ præcedit, dicitur eſſe *extra ſignum*, nempe 2 & a. A quibuſdam vero quantitates illæ vocantur *Coefficientes*. Cum quantitas nulla præcedit, ibi ſemper intelligitur unitas, ut $1\surd 2$, & $1\surd a$ &c.

IV. Ut quantitas extra ſignum ſub ſigno radicali poni poſſit, elevari prius debet ad illam ipſam poteſtatem, quam indicat exponens ſigni radicalis, & multiplicari per quantitatem ſub ſigno radicali exiſtentem. Sic $2\sqrt{3}$ fiet $\sqrt{12}$. Item $a\sqrt{b-c}$, erit $\sqrt{a^2b-a^2c}$. Demum $a\sqrt[m]{bc}$, erit $\sqrt[m]{a^mbc}$.

V. Contra vero quantitas ſub ſigno radicali exiſtens, & eundem exponentem habens, quem habet ſignum radicale, eſt quantitas rationalis, & ponitur extra ſignum extracta radice. Sic $\sqrt{ab^2}=b\sqrt{a}$, $\sqrt[3]{am^3}=m\sqrt[3]{a}$, $\sqrt[n]{a^nb}=a\sqrt[n]{b}$. Relinquitur ſub ſigno id, quod radicale eſt.

VI. Quantitas negativa ſub ſigno radicali, cujus exponens eſt numerus par, dicitur radix *imaginaria*, ſeu impoſſibilis, ut $\sqrt[2]{-a}$, $\sqrt[4]{-a}$, $\sqrt[6]{-a}$; ſimiliter $\sqrt[2]{-3}$, $\sqrt[4]{-5}$ &c. nam impoſſibile eſt quadratum negativum ex dictis in *Schol. 1. Prop. 8. Cap. 3.*

VII. Duæ quantitates radicales dicuntur *commenſurabiles* inter ſe, ſeu *communicantes*, cum ad ſimplicem expreſſionem redactæ, habent ſub ſigno radicali quantitatem eandem, ut $3\sqrt{5}$ & $2\sqrt{5}$; pariter $b\sqrt{a-b}$ & $c\sqrt{a-b}$. Nam exprimi poteſt ratio, quam inter ſe habent; eſt enim $3\sqrt{5} \,.\, 2\sqrt{5} :: 3 \,.\, 2$. Item $b\sqrt{a-b} \,.\, c\sqrt{a-b} :: b \,.\, c$, ut patet.

VIII. Si

VIII. Si radicale ſignum quodcunque comprehendat ſub ſe aliam quantitatem radicalem, ut $\sqrt{2+\sqrt{3}}$, vel $\sqrt[3]{a^2 - b\sqrt{c}}$, radix *univerſalis* dicitur; & denominatur a ſigno, quod totam illam quantitatem, linea ſuperducta, ſeu vinculo, complectitur. Sic $\sqrt[3]{2+3\sqrt{5}}$ dicitur radix univerſalis cubica: hoc eſt ex quantitate $2+3\sqrt{5}$ radicem cubicam eſſe extrahendam eo pacto, quo inferius docebitur.

PROPOSITIO I.

Quantitates radicales diverſæ denominationis ad eandem reducere.

I. Reducendæ ſint radicales $\sqrt[2]{a}$ & $\sqrt[3]{b}$ ad eandem denominationem. Erit ex *Defin. 5. Cap. 3.* $\sqrt{a} = a^{\frac{1}{2}}$ & $\sqrt[3]{b} = b^{\frac{1}{3}}$, redactiſque fractionibus ex communi Arithmetica ad eandem denominationem, fiunt $a^{\frac{3}{6}}$ & $b^{\frac{2}{6}}$; proinde a elevari debet ad tertiam poteſtatem, & b ad ſecundam, ut indicant novi numeratores fractionum. Signis autem radicalibus præfigendus exponens 6, communis ſcilicet ipſarum fractionum denominator. Erunt ergo $\sqrt[6]{a^3}$ & $\sqrt[6]{b^2}$.

Similiter reducendæ ſint $\sqrt[2]{2}$ & $\sqrt[6]{20}$ ad eandem denominationem. Quia $\sqrt{2} = 2^{\frac{1}{2}}$ & $\sqrt[6]{20} = 20^{\frac{1}{6}}$ *per Defin. cit.*

cit. redactis fractionibus $\frac{1}{2}$ & $\frac{1}{6}$ ad idem nomen, habentur $\frac{6}{12}$ & $\frac{2}{12}$, quibus ad minores terminos reductis, dividendo per 2, habentur $\frac{3}{6}$ & $\frac{1}{6}$; unde patet 2 ad tertiam poteſtatem eſſe elevandum, & 20 ad primam; erunt ergo $\sqrt[6]{8}$ & $\sqrt[6]{20}$.

II. Vel ſic: reducendæ ſint ad idem ſignum $\sqrt[2]{c}$ & $\sqrt[6]{a}$, quia exponens 6 præciſe diviſibilis eſt per exponentem 2, multiplico per quotum 3 tam exponentem 2, quam quantitatem ſub ſigno exiſtentem c, eam elevando ad potentiam tertiam, eritque $\sqrt[6]{c^3}$ ejuſdem nominis cum $\sqrt[6]{a}$. Eadem ratione $\sqrt[2]{3}$ & $\sqrt[4]{5}$ reductæ ad idem ſignum, fiunt $\sqrt[4]{9}$ & $\sqrt[4]{5}$. Praxis oritur ex *Coroll. Prop.* 3. *Cap.* 2.

Ratio autem eſt, quia una radix non fit major, etiamſi ejus potentia major fiat, hoc eſt ad altiorem gradum elevetur. Eſt enim a radix tam potentiæ a^2, quam potentiæ a^3, a^4, a^5 &c. proinde ſi $\sqrt{a^2}$ per multiplicationem fiat $\sqrt[3]{a^3}$ remanet idem valor, & $\sqrt[2]{a^2} = \sqrt[3]{a^3}$.

SCOLL. I. *In reducendis ad eandem denominationem, ſive ad idem ſignum radicalibus, quantitates, quæ ſunt extra ſignum, non immutantur, ſed manent eædem. Reducendæ ſint ad idem ſignum* $3\sqrt[2]{2}$ *&* $4\sqrt[3]{5}$, *fient* $3\sqrt[6]{8}$ *&* $4\sqrt[6]{25}$. *Similiter* $a\sqrt[2]{b}$ *&* $c\sqrt[3]{n}$, *fient* $a\sqrt[6]{b^3}$ *&* $c\sqrt[6]{n^2}$. *Nam* $a\sqrt[2]{b} = \sqrt[2]{a^2 b}$ per Defin. 4. *hujus*, *ſed* $\sqrt[2]{a^2 b} = \sqrt[6]{a^6 b^3}$ per

per hanc Propoſ., & $\sqrt[6]{a^6b^3}=a\sqrt{b^3}$ per Prop. 7. Cap. 3. *Ergo* $a\sqrt[2]{b}=a\sqrt[6]{b^3}$. *Eadem ratione* $c\sqrt[3]{n}=\sqrt[3]{c^3 n}=\sqrt[6]{c^6 n^2}=c\sqrt[6]{n^2}$, *adeoque radicalium coefficientes non mutantur.* *Demonſtrationem licet per ſe claram tyrones ex dicendis magis intelligent.*

SCHOL. II. *Ut radicales addi, ſubtrahi, multiplicari, aut dividi invicem poſſint, eas ad idem ſignum prius reducere neceſſe eſt, quod notetur.*

PROPOSITIO II.

Radicales ad ſimpliciorem expreſſionem reducere.

1. DIvidatur quantitas ſub ſigno exiſtens per poteſtatem ejuſdem gradus, ſeu ejuſdem exponentis, quem præfert ſignum radicale, & relicto quoto ſub ſigno radix poteſtatis, per quam facta fuit diviſio, ponatur extra ſignum: hoc eſt extrahatur a quantitate radicali id, quod eſt rationale, & ponatur extrà ſignum. Sit $\sqrt{aac+aab}$ divide per a^2, & quoto $c+b$ ſub ſigno relicto, pone extra ſignum radicem poteſtatis dividentis, nempe a, erit radicalis ad ſimpliciorem expreſſionem redacta $a\sqrt{c+b}=\sqrt{aac+aab}$.

Sit pariter $\sqrt[3]{54}$ reducenda ad ſimpliciorem expreſſionem. Divide 54 per cubum 27, & relicto quoto 2 ſub ſigno, pone extra ſignum radicem ejuſdem cubi, nempe 3,

erit $3\sqrt[3]{2} = \sqrt[3]{54}$. Eadem ratione $\sqrt[3]{8a^3b}$ erit $2a\sqrt[3]{b}$, & $\sqrt[n]{ab^n}$ erit $b\sqrt[n]{a}$.

II. Quod ſi quantitas radicalis ſit valde compoſita, ita ut ex ſola terminorum inſpectione non innoteſcat, an per maximam ejuſdem gradus poteſtatem ſit diviſibilis, quærantur *per Prop.* 9. *Cap.* 1. omnes illius diviſores: ex illis neceſſario apparebit, an ſit poteſtas requiſita. Quæritur ex. gr. an $\sqrt[3]{120}$ ſit reducibilis ad ſimpliciorem expreſſionem. Inter ejus diviſores apparet 8, qui eſt tertia poteſtas requiſita. Divide ergo 120 per 8, & relicto quoto 15 ſub ſigno, habebis $2\sqrt[3]{15} = \sqrt[3]{120}$. Item ſi $\sqrt[4]{48}$ reducenda ſit ad ſimpliciorem, erit $2\sqrt[4]{3}$, nam inter diviſores numeri 48, eſt 16, qui eſt poteſtas quarta quæſita, ac ter dividit ipſum 48.

COROL. I. *Pro faciliori praxi, divide quantitatem radicalem ſub ſigno exiſtentem per* 2, *per* 3, *per* 4 *&c. ſucceſſive, & examina, an quoti illi ſint poteſtas quæſita: quod facile agnoſces ſi ex illis radicem extrahas. Sit* $\sqrt{392}$. *divide* 392 *per* 2, *quotus* 196 *eſt quadratum ex radice* 14, *per quod ſi dividas* 392, *habebis* 2, *proinde* $\sqrt{392} = 14\sqrt{2}$. *Similiter ſit reducenda* $\sqrt[3]{192}$, *divide per* 3, *quotus* 64 *dat cubum ex latere* 4, *per quem cubum dividendo* 192, *habebis* 3; *ideoque* $\sqrt[3]{192} = 4\sqrt[3]{3}$, *& ſic de aliis.*

COROLL. II *Ex hac propoſitione dignoſcitur, an duæ radicales ejuſdem gradus ſint inter ſe commenſurabiles &*

commu-

communicantes. Nam ſi ſcire velim, an $\sqrt{8}$ *&* $\sqrt{18}$ *ſint communicantes, reduco illas ad ſimpliciorem expreſſionem* $2\sqrt{2}$ *&* $3\sqrt{2}$, *quæ* ex Defin. 6. hujus *erunt inter ſe, ut* 2 *ad* 3. *Ita* $\sqrt{50}$ *&* $\sqrt{18}$ *redactæ, fiunt* $5\sqrt{2}$ *&* $3\sqrt{2}$; *erit ergo* $\sqrt{50}.\sqrt{18}::5.3$.

SCHOL. *Ut id facile obtineatur, ex quantitatibus ſub ſigno radicali exiſtentibus fiat fractio, quæ ad minimos terminos reducatur. Sic in ſuperiori exemplo* $\sqrt{8}$ *&* $\sqrt{18}$ *facile dignoſcuntur eſſe commenſurabiles. Nam fractio* $\frac{8}{18}$ *ad minimos terminos redacta fit* $\frac{4}{9}$. *Sunt ergo* $\sqrt{8}.\sqrt{18}::\sqrt{4}.\sqrt{9}::2.3$. *Similiter ex* $\sqrt{50}$ *&* $\sqrt{18}$ *fiat fractio* $\frac{50}{18}$, *quæ reducitur ad minimos terminos* $\frac{25}{9}$; *unde* $\sqrt{50}.\sqrt{18}::\sqrt{25}.\sqrt{9}::5.3$. *Demum* $\sqrt[3]{a^3 b}$ *&* $\sqrt[3]{bc^3}$ *eadem ratione inveniuntur eſſe ut* a *ad* c.

PROPOSITIO III.

Quantitates radicales addere, aut unam ex altera ſubtrahere.

I. RAdicales addendæ, aut ſubtrahendæ reducantur, ſi fieri poteſt, ad ſimpliciorem expreſſionem *per Prop. antec.* Si communicantes ſint, quantitates rationales extra ſignum addantur, vel ſubtrahantur, ut fieri ſolet in quantitatibus rationalibus, & earum ſumma, aut differentia præponitur ſigno radicali. Sint addendæ quantitates $\sqrt{50}$ & $\sqrt{18}$, quæ redactæ *per Prop. cit.* ſunt $5\sqrt{2}$ & $3\sqrt{2}$, additis quantitatibus extra ſignum 5 & 3, earum ſumma erit $8\sqrt{2}$. Subtrahenda ſit $\sqrt{18}$ ex $\sqrt{50}$, hoc eſt $3\sqrt{2}$ ex $5\sqrt{2}$ *per prop. antec.* ſubtrahe 3 ex 5, reſiduum erit $2\sqrt{2}$

II. Quod ſi radicales addendæ, aut ſubtrahendæ non ſint communicantes, ſed incommenſurabiles, ſumma fit per ſignum +, ſubtractio per ſignum —. Sic duarum $\sqrt{2}$ & $\sqrt{5}$ ſumma erit $\sqrt{2}+\sqrt{5}$; differentia vero $\sqrt{2}-\sqrt{5}$. Similiter duarum $\sqrt{a+b}$ & $\sqrt{a-b}$ ſumma erit $\sqrt{a+b}+\sqrt{a-b}$; differentia autem $\sqrt{a+b}-\sqrt{a-b}$.

III. Quod ſi radicales addendæ, vel ſubtrahendæ ſint compoſitæ, eadem ratione operandum, ut exempla ſequentia indicant.

Additionis Compoſitæ Exempla.

$$\begin{array}{r} 3\sqrt{2}-5\sqrt{3}+6\sqrt{5}+2\sqrt{6} \\ 2\sqrt{2}+3\sqrt{3}-8\sqrt{5}+\sqrt{6} \\ \hline 5\sqrt{2}-2\sqrt{3}-2\sqrt{5}+3\sqrt{6} \end{array}$$

Subtractionis Compoſitæ Exempla.

$$\begin{array}{r} 4\sqrt{3}-4\sqrt{2}+5\sqrt{6}-3\sqrt{7} \\ 2\sqrt{3}+2\sqrt{2}-7\sqrt{6}+5\sqrt{7} \\ \hline 2\sqrt{3}-6\sqrt{2}+12\sqrt{6}-8\sqrt{7} \end{array}$$

PROPOSITIO IV.

Quantitates radicales multiplicare.

I. Multiplicentur inter ſe quantitates extra ſignum, tum ſeorſim quantitates ſub ſigno exiſtentes, habe-

habebis productum quæſitum, ut $5\sqrt{3}\times 4\sqrt{2}=20\sqrt{6}$. Similiter $m\sqrt[3]{a}\times n\sqrt[3]{a}=mn\sqrt[3]{aa}$.

II. Quod ſi radicales habeant ſignum radicale quadraticum $\sqrt{}$, & ſint communicantes, operatio fit brevior. Nam ſatis eſt productum, quod fit ex quantitatibus extra ſignum exiſtentibus, ducere in quantitatem ſub ſigno radicali poſitam. Sint multiplicandæ $3\sqrt{2}$ & $5\sqrt{2}$. Duc primo inter ſe 3 & 5, earumque factum 15 duc in 2, erit productum 30. Oritur enim quantitas rationalis; nam $\sqrt{2}\times\sqrt{2}=\sqrt{4}=2$, adeoque $15\times 2=30$.

Sint pariter multiplicandæ $\sqrt{x^4+x^2a^2}$ & $\sqrt{x^2a^2+a^4}$, quæ ad ſimpliciores terminos redactæ *per Prop.* 2. *hujus*, ſunt $x\sqrt{x^2+a^2}$, & $a\sqrt{x^2+a^2}$, proinde communicantes *per Defin.* 6. Duc itaque productum $ax\times\overline{x^2+a^2}$, in quantitatem ſcilicet ſub ſigno exiſtentem perinde ac forent quantitates rationales, habebis productum rationale ax^3+a^3x.

III. Si radicalis per rationalem, aut rationalis per radicalem ſit multiplicanda (idem enim eſt) ex.gr. $\sqrt{a+b}\times\overline{c+d}$. Reducatur primo rationalis $c+d$ ad idem ſignum radicale, eam ſcilicet elevando ad poteſtatem ejuſdem gradus, ita ut fiat $\sqrt{c^2+2cd+d^2}$. Multiplicetur deinde $c^2+2cd+d^2$ per $a+b$, perinde ac eſſent quantitates rationales, erit productum quæſitum $\sqrt{ac^2+2acd+ad^2+bc^2+2bcd+bd^2}$, quod etiam exprimi poterat hoc modo $\overline{c+d}\sqrt{a+b}$.

Multiplicando pariter $3\sqrt[n]{5}$ per 2, aut per $\frac{1}{2}$, productum erit $6\sqrt[n]{5}$, aut $\frac{3}{2}\sqrt[n]{5}$.

IV. Si quantitates multiplicandæ compoſitæ ſint ex radicalibus, vel ex radicalibus ſimul & rationalibus, ducantur ſinguli termini unius per ſingulos alterius ſecundum regulas radicalium ſimplicium ſuperius explicatas, ſervata conſueta ſignorum + & — ratione. Exempla rem ſatis declarant.

I.
$$\begin{array}{r} \sqrt{5}+\sqrt{3} \\ \sqrt{5}-\sqrt{3} \\ \hline 5+\sqrt{15} \qquad \\ -\sqrt{15}-3 \\ \hline 5-3=2 \end{array}$$

II.
$$\begin{array}{r} 3+2\sqrt{5} \qquad \\ 5-3\sqrt{5} \qquad \\ \hline 15+10\sqrt{5} \qquad \\ -9\sqrt{5}-30 \\ \hline +\sqrt{5}-15 \end{array}$$

III.
$$\begin{array}{l} 2a+\sqrt{a+x} \\ b-\sqrt{a+x} \\ \hline 2ab+b\sqrt{a+x} \\ \quad -2a\sqrt{a+x}-a+x \\ \hline 2ab+b\sqrt{a+x}-2a\sqrt{a+x}-a+x \end{array}$$

Ratio multiplicandi quantitates radicales ita generatim demonſtratur. Sit multiplicanda $\sqrt{3}$ per $\sqrt{2}$, dico productum eſſe $\sqrt{6}$. Nam ex definitione multiplicationis ita debet eſſe unitas ad multiplicantem ($\sqrt{2}$), ut multiplicandus

dus ($\sqrt{3}$) ad productum, quod voco P; erit ergo $1 . \sqrt{2} :: \sqrt{3} . P$. & in eadem ratione erunt eorum quadrata, nempe $1 . 2 :: 3 . P^2$, ſed $1 . 2 :: 3 . 6$, ergo $P^2 = 6$, & $P = \sqrt{6}$. Quod erat &c.

COROLL. I. *Quod ſi binomium radicale quadraticum multiplicetur per ſe ipſum, mutato tamen in alterutro factore ſigno + in —, vel contra — in +, oritur monomium rationale, ut in primo exemplo* $\sqrt{5} + \sqrt{3} \times \sqrt{5} - \sqrt{3}$ *producit quantitatem rationalem 2, & hoc dicitur binomium multiplicari* per ſui contrarium, *de quo inferius.*

COROLL. II. *Si trinomium radicale quadraticum, ex. gr.* $\sqrt{6} + \sqrt{2} + \sqrt{5}$ *multiplicetur per ſe ipſum, mutato aliquo ex ſignis ut per* $\sqrt{6} + \sqrt{2} - \sqrt{5}$, *oritur binomium partim radicale, partim rationale, ut* $3 + \sqrt{48}$, *quod per* Coroll. 1. *ad monomium omnino rationale facile reducitur.*

SCHOL. *Si radicales ſignum diverſum habeant, prius reducentur ad idem ſignum ex dictis* Schol. 3. Prop. 1. *hujus, deinde fiet multiplicatio ut ſupra. Sint multiplicandæ* $\sqrt[3]{5} \times \sqrt[3]{7}$, *erunt reductæ* per Propoſ. 1. *hujus* $\sqrt[6]{125}$ *&* $\sqrt[6]{49}$, *earumque productum* $\sqrt[6]{6125}$.

PROPOSITIO V.

Quantitates radicales dividere.

I. SI diviſor & dividendus ſint communicantes, ſatis eſt dividere quantitates extra ſignum, quotus eſt quantitas rationalis. Sic ad dividendum $6\sqrt{3}$ per $3\sqrt{3}$, divi-

diviſo 6 per 3, quotus eſt 2. Similiter $4\sqrt{2}$ diviſo per $2\sqrt{2}$, habetur quotus 2. Eadem ratione $a\sqrt{a+b}$ diviſo per $b\sqrt{a+b}$, habetur pro quoto $\frac{a}{b}$.

Quod ſic demonſtratur. Ponantur ſub ſigno quantitates 6 & 3: erit $6\sqrt{3}=\sqrt{108}$ & $3\sqrt{3}=\sqrt{27}$. Dividatur jam $\sqrt{108}$ per $\sqrt{27}$, hoc eſt 108 per 27, quotus eſt $\sqrt{4}=2$. Quod &c. Similiter poſitis ſub ſigno a & b, erit $a\sqrt{a+b}=\sqrt{a^3+a^2b}$, & $b\sqrt{a+b}=\sqrt{ab^2+b^3}$, & dividendo $\sqrt{a^3+a^2b}$ per $\sqrt{ab^2+b^3}$ (hoc eſt a^3+a^2b per ab^2+b^3, ut fit in rationalibus) erit quotus $\frac{\sqrt{aa}}{\sqrt{bb}}=\frac{a}{b}$. Quod &c.

II. Si diviſor & dividendus non fuerint communicantes, dividantur ſeorſim quantitates extra ſignum & ſeorſim, quæ ſub ſigno exiſtunt. Sic ad dividendum $6\sqrt{ab}$ per $2\sqrt{a}$ habetur quotus $3\sqrt{b}$. Item dividendo $\sqrt{6}$ per $2\sqrt{3}$, quotus erit $\frac{1}{2}\sqrt{2}$. Demum $\sqrt{15}$ per $\sqrt{3}$ dat quotum $\sqrt{5}$.

III. Quod ſi dividenda ſit quantitas radicalis per rationalem, vel e contrario: ſemper quantitas rationalis ad radicalem reducatur, eam ſcilicet elevando ad poteſtatem ejuſdem gradus, & ponendo illam ſub ſigno radicali. Ut ſi dividenda ſit a per $\sqrt[3]{ab}$, elevetur a ad tertiam poteſtatem ita ut fiat $\sqrt[3]{a^3}$, quæ diviſa per $\sqrt[3]{ab}$ dabit quotum

tum $\frac{\sqrt[3]{aa}}{\sqrt[3]{b}} = \sqrt[3]{\frac{aa}{b}}$, ut in ſequenti *Schol. I. hujus.*

Ratio dividendi radices ita generatim demonſtratur. Dividenda ſit radicalis $\sqrt{15}$ per $\sqrt{3}$, dico quotum eſſe $\sqrt{5}$. Nam ex communi diviſionis definitione, diviſor ad dividendum eſt, ut unitas ad quotum, quem voco Q: hoc eſt, $\sqrt{3} . \sqrt{15} :: 1 . Q$; & in eadem ratione ſunt eorum quadrata, $3 . 15 :: 1 . Q^2$. Sed $3 . 15 :: 1 . 5$, ut patet; proinde $Q^2 = 5$, & $Q = \sqrt{5}$. Quod erat &c.

SCHOL. I. *Si radicalium diviſio exprimatur per modum fractionis (quod non raro fit) tunc ſignum radicale, vel ſeorſim numeratori, & ſeorſim denominatori præponitur, ut ſuperiora exempla docent; vel toti fractioni, ſcilicet* $\sqrt{\frac{a}{b}}$, $\sqrt[3]{\frac{a^2}{bc}}$ *&c.*

SCHOL. II. *Si ex denominatore tantum, vel ex ſolo numeratore extrahi poſſit radix, utique extrahitur, ſed alteri ſignum radicale præfigitur. Sit* $\frac{\sqrt{axx}}{\sqrt{ab}}$, *facta diviſione erit* $\frac{\sqrt{xx}}{\sqrt{b}}$, *extractaque ex numeratore radice, fit* $\frac{x}{\sqrt{b}}$. *Similiter ex* $\sqrt{\frac{cb}{a^2b}}$ *facta diviſione, oritur* $\sqrt{\frac{c}{a^2}}$, *& extracta ex denominatore radice, habetur* $\frac{\sqrt{c}}{a}$.

PROPOSITIO VI.

Radicales compoſitas dividere.

I. Dividenda ſit radicalis compoſita $\surd 15 + \surd 21 - \surd 27$ per $\surd 3$. Dividantur ſingula trinomii membra per $\surd 3$, ut radicales ſimplices *per Prop. præc.* erit quotus $\surd 5 + \surd 7 - 3$. Eadem ratione dividendo per $\surd 5$ radicalem $\surd 20 - \surd 10 + \surd 15$, erit quotiens $2 - \surd 2 + \surd 3$. Rurſus dividendum ſit trinomium $10 \surd 2 + 2 \surd 15 - 3 \surd 35$ per 4, erit quotus $\frac{5}{2} \surd 2 + \frac{1}{2} \surd 15 - \frac{3}{4} \surd 35$.

II. Quod ſi dividere oporteat binomium *A* ex rationali quantitate & radicali compoſitum per aliud binomium ſimile *B*, diviſa quantitate rationali 8 per rationalem 2, habetur quotus *C*, per quem multiplicando totum diviſorem *B*, productum $8 + 4 \surd 1$ ſubducitur (mutatis ſignis) ex dividendo *A*, & nihil remanet. Idem fit in ſecundo exemplo. In tertio autem, per eundem diviſorem *N* dividitur fractionis *M* tam numerator, quam denominator, ut habeatur fractio in minimis terminis, ut in *Q*, quo exemplo infra utemur.

	B	*A*	*C*
EXEMPL. I.	$2 + \surd 1)$	$8 + 4 \surd 1$	$(+4$
		$\underline{8 + 4 \surd 1}$	
		$0 \quad 0$	

EXEMPL. II.	$-1 + \surd -\frac{3}{4})$	$2 - 2 \surd - \frac{3}{4}$	$(-2$
		$\underline{2 - 2 \surd - \frac{3}{4}}$	
		$0 \quad 0$	

EXEM-

EXEMPLUM III.

$$N\ aa+cc)\ .\ \overset{M}{\frac{a^3\sqrt{aa+cc}+acc\sqrt{aa+cc}}{2aa+2cc}}\ \Big|\ \overset{Q}{\frac{a\sqrt{aa+cc}}{2}}$$

seu $\frac{1}{2}a\sqrt{aa+cc}$

III. At ſi diviſor ſit binomium radicale ex.gr. $\sqrt{5}-\sqrt{2}$, per quod dividenda ſit quantitas $\sqrt{18}$; tunc binomium illud reducatur ad monomium rationale *per Cor. 1. Prop. 4. hujus*, multiplicando illud per ſui contrarium $\sqrt{5}+\sqrt{2}$, habebitur novus diviſor $5-2=3$. Deinde per idem contrarium $\sqrt{5}+\sqrt{2}$ multiplicetur quoque quantitas dividenda, nempe $\sqrt{18}$, ut inter ipſam & diviſorem maneat idem valor, qui antea, *per Axiom. 2. Cap. 2.* Exemplo res ſit manifeſta.

Diviſor $\sqrt{5}-\sqrt{2}$	Dividen. $+\sqrt{18}$
$\sqrt{5}+\sqrt{2}$	$\sqrt{5}+\sqrt{2}$
$5-\sqrt{10}$	3) $\sqrt{90}+6$
$+\sqrt{10}-2$	Quotus $\frac{1}{3}\sqrt{90}+2$
$5-2=3$	

Ratio clara eſt, ſi ex diviſore & dividendo fiat fractio, & tam numerator quam denominator multiplicentur per eandem quantitatem. Sit enim dividenda radicalis quantitas *A* per trinomium radicale *B*; fiat fractio, nempe

$$A\ \frac{\sqrt{5}}{\sqrt{3}+\sqrt{1}+\sqrt{2}}$$

Multiplicato utroque fractionis termino per denominatorem B (uno tantum ſigno mutato), oritur nova fractio æqualis priori *per Axiom. 2. Cap. 2.* hoc eſt

$$\frac{C}{D}\ \frac{\sqrt{15}+\sqrt{5}-\sqrt{10}}{2+\sqrt{12}}$$

Multiplicetur pariter uterque terminus hujus novæ fractionis per denominatorem D (mutato ſigno), oritur nova fractio priori æqualis *per Axiom. cit.*, in qua diviſor eſt quantitas rationalis, ſcilicet

$$\frac{2\sqrt{15}+2\sqrt{5}-2\sqrt{10}-\sqrt{180}-\sqrt{60}+\sqrt{120}}{-8}$$

Factaque actu diviſione, erit quotus

$$-\tfrac{1}{4}\sqrt{15}-\tfrac{1}{4}\sqrt{5}+\tfrac{1}{4}\sqrt{10}+\tfrac{1}{8}\sqrt{180}+\tfrac{1}{8}\sqrt{60}-\tfrac{1}{8}\sqrt{120}.$$

Item, ut dividatur $a-b$ per $\sqrt{a}-\sqrt{b}$, fit fractio $\frac{a-b}{\sqrt{a}-\sqrt{b}}$, cujus termini per $\sqrt{a}+\sqrt{b}$ multiplicati dant $\frac{\overline{a-b}\times\overline{\sqrt{a}+\sqrt{b}}}{a-b}$: fiat diviſio, erit quotus $\sqrt{a}+\sqrt{b}$.

Schol. I. *Si diviſor ſit binomium, aut trinomium conſtans radicalibus cubicis, aut altiorum poteſtatum, hæc regula non ſufficit. Tunc invenienda eſt aliqua quantitas, quæ* diviſor contrarius *dicitur, per quam ſi tale binomium, aut trinomium multiplicetur, fiat rationale. En methodus generalis inveniendi ejusmodi* diviſores contrarios. *Quantitates ſub ſigno radicali exiſtentes, mutato aliquo ſigno, eleventur ad poteſtatem, cujus exponens ſit unitate minor exponente radicali, neglectisque coefficientibus, habetur quantitas quæſita.*

ſita. Sit binomium $\sqrt[3]{a}+\sqrt[3]{b}$, *elevetur* $a-b$ (*mutato ſcilicet ſigno*) *ad poteſtatem ſecundam, quæ neglectis coefficientibus, erit* a^2-ab+b^2, *eaque poſita ſub ſignis radicalibus dati binomii, habetur* diviſor contrarius $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$.

Similiter ſit datum binomium $\sqrt[4]{a}+\sqrt[4]{b}$, *elevetur* $a-b$ *ad tertiam poteſtatem, & abjiciantur coefficientes, oritur quantitas* $a^3-a^2b+ab^2-b^3$, *qua ſub ſignis radicalibus binomii poſita, habetur* diviſor contrarius $\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}$; *& ſic de aliis.*

SCHOL. II. *Ceterum hujuſmodi diviſiones radicalium compoſitarum, ut monet* Cl. Wolfius (a), *aut vix, aut perraro occurrunt. Sed quia ejus methodus, quam* Nicolao Tartaleæ (b) *debemus, eſt non inelegans, haud erat omnino negligenda.*

PROPOSITIO VII.

Radicales ad datam poteſtatem elevare.

I. Regula eſt: quantitates ſub ſignis exiſtentes ad datam poteſtatem attolluntur *per Prop. 2., vel 3. Cap. 3.* radicali ſigno minime variato. Sic radicalis $\sqrt[3]{a}$, elevata ad

 quadra-

(a) Element. Analyt. edit. 2. Probl. 13. Schol. 1.
(b) De Num. & Menſuris part. 2. l. 5. cap. 4.

quadratum erit $\sqrt[3]{a^2}$. Radicalis $\sqrt[2]{b}$ elevata ad cubum erit $\sqrt[2]{b^3}$. Item $\sqrt[2]{3}$ elevata ad quartam poteſtatem fit $\sqrt[2]{81}$.

Ratio eſt, quia ex communi lege multiplicationis radicalium, $\sqrt[3]{a}\times\sqrt[3]{a}=\sqrt[3]{a^2}$. Similiter $\sqrt[2]{b}\times\sqrt[2]{b}\times\sqrt[2]{b}=\sqrt[2]{b^3}$ *per Prop. 4. hujus.*

II. Si adſit quantitas extra ſignum, illa quoque ad datam poteſtatem elevatur *per Prop. 2., vel 3. Cap. 3.* Sic radicalis $a\sqrt[3]{c}$ ad quadratum elevata, erit $a^2\sqrt[3]{c^2}$. Nam $a\sqrt[3]{c}\times a\sqrt[3]{c}=a^2\sqrt[3]{c^2}$. Eadem ratione $2\sqrt[2]{a}$ ad cubum evecta erit $8\sqrt[2]{a^3}$. Nam $2\sqrt[2]{a}\times 2\sqrt[2]{a}\times 2\sqrt[2]{a}=8\sqrt[2]{a^3}$ *per Prop. 4. hujus.*

III. Si radicales ad datam poteſtatem elevandæ ſint complexæ, elevantur ad illam eadem omnino ratione, qua elevantur quantitates rationales; hoc eſt, ſi elevandæ ſint ad quadratum, ſumi debent quadrata partium, & duplum ejus, quod oritur ex mutua earundem partium multiplicatione *per Cor. Prop. 8. Cap. 3.* Si ad cubum, ſumi debent cubi partium, triplum ejus, quod oritur multiplicando quadratum primæ per ſecundam, & triplum ejus, quod oritur multiplicando viciſſim quadratum ſecundæ per primam *per Cor. Prop. 9. Cap. 3.* Atque ita deinceps. Sic radicalis $\sqrt[3]{a}+\sqrt[3]{c}$, ut ad ſecundam poteſtatem elevetur, fiant per primam partem *Prop. hujus* quadrata partium, nempe $\sqrt[3]{a^2}$ & $\sqrt[3]{c^2}$ earumque factum duplum $2\sqrt[3]{ac}$, proinde

inde erit $\sqrt[3]{a^2}+2\sqrt[3]{ac}+\sqrt[3]{c^2}$. Similiter $\sqrt{a}-\sqrt{b}$ ad ſecundam poteſtatem evecta erit $a+b-2\sqrt{ab}$. Si vero $\sqrt{a}+\sqrt{b}$ elevanda ſit ad cubum, fiant primo per primam partem *Propoſ. hujus* cubi ex $\sqrt{a}$ & $\sqrt{b}$, nempe $\sqrt{a^3}$ & $\sqrt{b^3}$; ſumatur triplum ex multiplicatione quadrati primæ radicalis, nempe a, in ſecundam, hoc eſt $3a\sqrt{b}$, & viciſſim triplum quadrati ſecundæ, nempe b, ducti in primam, hoc eſt $3b\sqrt{a}$, iisque additis, erit $\sqrt{a^3}+3a\sqrt{b}+3b\sqrt{a}+\sqrt{b^3}$. Eadem ratione $\sqrt{3}-\sqrt{2}$ ad cubum evecta, erit $\sqrt{27}-9\sqrt{2}+6\sqrt{3}-\sqrt{8}$

SCHOL. I. *Cum radicales ſimplices conſiderari poſſint, ut potentiæ imperfectæ* per Defin. 5. Cap. 3. *elevari poſſunt ad quamcunque datam poteſtatem, ſumendo duplum, triplum, quadruplum &c. earum exponentium, ſi ad ſecundam, ad tertiam, vel quartam poteſtatem elevandæ ſint. Nam ſit* $\sqrt[2]{a}$ *ad tertiam poteſtatem elevanda*, per Defin. cit. *eſt* $\sqrt[2]{a}=a^{\frac{1}{2}}$, *& ſumendo triplum bujus exponentis, erit* $a^{\frac{3}{2}}=\sqrt[2]{a^3}$. *Hoc autem ſequitur* ex Prop. 1. Cap. 3. *Nam* $a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}=a^{\frac{3}{2}}$.

SCHOL. II. *Si radicalis elevanda ſit ad poteſtatem, cujus exponens idem ſit cum exponente ejusdem radicalis, oritur quantitas rationalis. Proinde ſatis eſt delere ſignum radicale. Nam* $\sqrt[2]{3}\times\sqrt[2]{3}=\sqrt[2]{9}=3$. *Sic etiam* $\sqrt[3]{2}$ *ad cubum evecta erit* 2. *Nam* $\sqrt[3]{2}\times\sqrt[3]{2}\times\sqrt[3]{2}=\sqrt[3]{8}=2$.

Quod

Quod etiam ex Schol. 1. *infertur. Nam* $\sqrt[2]{a}=a^{\frac{1}{2}}$, *proinde* $a^{\frac{1}{2}+\frac{1}{2}}=a^{\frac{2}{2}}=a$. *Pariter* $\sqrt[3]{c}=c^{\frac{1}{3}}$, *quæ ad tertiam poteſtatem evecta ſit* $c^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=c^{\frac{3}{3}}=c$.

PROPOSITIO VIII.

Ex ſimplici radicali radicem extrahere.

I. Extrahenda ſit radix quadrata ex $\sqrt{a}$, quia *per Defin. 5. Cap. 3.* $\sqrt{a}=a^{\frac{1}{2}}$, dividatur per exponentem radicis quæſitæ exponens ipſius $a^{\frac{1}{2}}$, erit radix quadrata quæſita $a^{\frac{1}{4}}=\sqrt[4]{a}$.

Similiter extrahenda ſit radix quadrata ex $\sqrt[3]{a}$, quia *per Defin. cit.* $\sqrt[3]{a}=a^{\frac{1}{3}}$, dividendo per exponentem radicis quæſitæ exponentem ejusdem $a^{\frac{1}{3}}$, erit radix quæſita $a^{\frac{1}{6}}=\sqrt[6]{a}$.

II. Demum quæritur radix cubica quantitatis $\sqrt{a}$, quia $\sqrt{a}=a^{\frac{1}{2}}$ ſi dividatur per exponentem radicis quæſitæ 3 exponens ipſius $a^{\frac{1}{2}}$, erit radix cubica $a^{\frac{1}{6}}=\sqrt[6]{a}$.

Co-

COROLL. *Hinc patet, ad extrahendas ejuſmodi radices ſatis eſſe multiplicare exponentem ſigni radicalis per exponentem radicis quæſitæ. Sic radix quinta quantitatis* $\sqrt[2]{2}$ *erit* $\sqrt[10]{2}$. *Vel etiam exprimi poteſt ſic* $\sqrt[5]{\sqrt{2}}$, $\sqrt[3]{\sqrt{2}}$, $\sqrt{\sqrt{2}}$ &c.

PROPOSITIO IX.

Ex binomio radicem quadratam extrahere.

I. EXtrahenda ſit radix quadrata ex binomio dato $7 + \sqrt{48}$. Primo auferatur quadratum minoris termini ex quadrato majoris, hoc eſt 48 ex 49 quadrato, quod fit ex termino rationali 7. Secundo extrahatur ex differentia horum quadratorum (quæ hic eſt 1) radix quadrata, nempe 1, quam adde termino rationali, habebis 8, & ab eodem ſubtrahe, habebis 6. Summæ hujus & differentiæ ſemiſſes, hoc eſt 4 & 3 dant radicem quadratam dati binomii, nempe $\sqrt{4} + \sqrt{3}$, ſeu $2 + \sqrt{3}$.

Similiter quæritur radix quadrata binomii $8 - \sqrt{60}$. Differentia quadratorum eſt 4, ejus radix quadrata 2. Hanc adde termino rationali, habebis 10: ſubtrahe ab eodem, habebis 6. Semiſſes ipſorum 10 & 6 dant radicem quæſitam $\sqrt{5} - \sqrt{3}$.

II. Sit pariter extrahenda radix quadrata ex binomio $a + b - 2\sqrt{ab}$: ex quadrato $a^2 + 2ab + b^2$ primi termini aufer $4ab$ quadratum ſecundi termini $- 2\sqrt{ab}$, erit

 diffe-

differentia $a^2 - 2ab + b^2$, cujus radix quadrata eſt $a - b$. Hanc adde termino rationali $a + b$, ſumma erit $2a$: ſubtrahe ex eodem termino, differentia erit $2b$. Horum ſemiſſes a & b dant radicem quæſitam $\sqrt{a} - \sqrt{b}$.

Ex hoc tertio exemplo facile eruitur demonſtratio. Nam ſi binomium $\sqrt{a} - \sqrt{b}$ elevetur ad ſecundam poteſtatem, habetur $a + b - 2\sqrt{ab}$; ubi apparent duo termini, alter rationalis, continens ſummam radicum $a + b$, alter irrationalis continens duplum facti earundem radicum. Ex quadrato autem $a^2 + 2ab + b^2$, quod fit ex termino rationali, ſi auferatur quadratum ſecundi termini, nempe $4ab$, reſiduum $a^2 - 2ab + b^2$ eſt pariter quadratum, cujus radix quadrata $a - b$ eſt differentia quadratorum a & b. Hæc igitur differentia $a - b$ addita ad eorum ſummam $a + b$ dat $2a$, duplum ſcilicet quadrati ex $\sqrt{a}$; eademque differentia $a - b$ ſubducta ex eorum ſumma $a + b$ relinquit $2b$, duplum quadrati alterius membri $\sqrt{b}$; proinde horum ſemiſſes a & b dant terminos radicis quæſitæ $\sqrt{a} - \sqrt{b}$.

Corol. *Hinc infertur tria requiri, ut radix quadrata extrahi poſſit ex binomio radicali quadratico. Primo binomium conſtare non poteſt ſolis radicalibus, ſed uno membro rationali & altero radicali. Nam ſive fiat quadratum ex duobus terminis radicalibus, ut* $\sqrt{2} + \sqrt{3}$, *ſive ex rationali & radicali, ut* $3 + \sqrt{5}$; *ſemper oritur binomium continens terminum rationalem & alterum radicalem. Secundo terminus rationalis, ſeu quadratum illius ſemper debet eſſe majus termino radicali, qui ab eo debet ſubtrahi. Tertio differentia quadratorum, quæ fiunt*

ex

ex termino rationali & radicali, debet eſſe quadratum. Hæc omnia ex allatis exemplis ſatis patent.

SCHOL: *De radice cubica, aliiſque altioris gradus ex dato binomio, aut polynomio extrahendis, commodiori loco agemus. Ceterum ſi radix quadrata defectu earum conditionum extrahi nequeat ex dato binomio, tunc ejus binomii radix exprimitur ſigno radicali, quod utrumque binomii terminum comprehendit. Sic radix quadrata binomii* $1 + \sqrt{3}$ *erit* $\sqrt{1 + \sqrt{3}}$. *Idem dicendum de cubica, aliiſque altioribus radicibus, ſi extrahi nequeant. Tunc enim radix cubica binomii* $2 - \sqrt{7}$ *erit* $\sqrt[3]{2 - \sqrt[2]{7}}$ *&c. quæ quidem radices ita connexæ, dicuntur* radices univerſales, *de quibus inferius.*

PROPOSITIO X.

Ex binomio, aut polynomio radicem quadratam per formulam Newtonianam extrahere.

I. DEſignet *A* majorem dati binomii partem. *B* vero partem minorem; erit ex Newtono in *Arithmetica Univerſali* $\frac{A+\sqrt{AA-BB}}{2}$ quadratum majoris partis quæſitæ radicis, & $\frac{A-\sqrt{AA-BB}}{2}$ quadratum partis minoris, quæ cum majori annectenda eſt cum ſigno ipſius *B*.

Extrahenda ſit radix ex binomio $5+\sqrt{24}$: ponatur $A=5$, & $B=\sqrt{24}$, erit $\sqrt{AA-BB}=1$, proinde $\frac{A+\sqrt{AA-BB}}{2}=\frac{5+1}{2}$ hoc eſt 3, quadratum majoris partis radicis; & $\frac{A-\sqrt{AA-BB}}{2}=\frac{5-1}{2}=2$, quadratum partis minoris. Horum radices dant radicem quæſitam $\sqrt{3}+\sqrt{2}$.

II. Rurſus extrahenda ſit radix ex binomio $rr+2x\sqrt{rr-xx}$. Ponatur $A=rr$, & $B=2x\sqrt{rr-xx}$, erit $AA-BB=r^4-4rrxx+4x^4$, cujus radix $rr-2x^2=\sqrt{AA-BB}$; proinde $\frac{A+\sqrt{AA-BB}}{2}=\frac{2rr-2xx}{2}=rr-xx$. Similiter $\frac{A-\sqrt{AA-BB}}{2}=xx$. Ex horum radicibus habetur dati binomii radix $x+\sqrt{rr-xx}$.

III. Sit polynomium $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$, ſeu in terminis ſimplicioribus $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$ *per Prop. 2. hujus*. Intelligatur diviſum in duo binomia; & ponatur $10+2\sqrt{6}=A$, & $2\sqrt{10}+2\sqrt{15}=B$, erit $\sqrt{AA-BB}=\sqrt{24}=2\sqrt{6}$ *per Prop. cit.* Nam ſi quadratum ſecundi termini $2\sqrt{10}+2\sqrt{15}$, quod in terminis ſimplicioribus eſt $100+40\sqrt{6}$, auferatur ex quadrato primi termini $10+2\sqrt{6}$, quod pariter in terminis ſimplicioribus eſt $124+40\sqrt{6}$, reſiduum erit 24, hoc eſt $2\sqrt{6}$ *per Prop. cit.* Addatur ergo hoc reſiduum $2\sqrt{6}$ ad pri-

ad primum terminum $10+2\sqrt{6}$, erit $\frac{A+\sqrt{AA-BB}}{2}$ $=\frac{10+4\sqrt{6}}{2}=5+2\sqrt{6}$. Ex hoc autem binomio ſi ex prima parte *hujus Propoſ.* eruatur radix, habebitur $\sqrt{3}+\sqrt{2}$. Auferatur jam reſiduum ipſum $2\sqrt{6}$ ex primo termino $10+2\sqrt{6}$, erit $\frac{A-\sqrt{AA-BB}}{2}=\frac{10}{2}-5$, cujus radix quadrata, hoc eſt $\sqrt{5}$, ſi priori jam inventæ addatur, erit polynomii propoſiti radix quæſita $\sqrt{3}+\sqrt{2}+\sqrt{5}$.

Demonſtratio hujus methodi eadem eſt, ac ſuperioris, ut conſideranti patet.

PROPOSITIO XI.

Radicales univerſales ad idem nomen, & ad ſimpliciorem expreſſionem reducere.

I. REducuntur ad idem nomen radicales univerſales eodem modo, quo aliæ radicales, ut in *Prop.* 1. *hujus*. Sed claritatis gratia reducendæ ſint ad eandem denominationem $\sqrt[2]{\overline{b+\sqrt[2]{cd}}}$ & $\sqrt[3]{\overline{a-\sqrt[3]{bc}}}$. Fractiones radicalium exponentiales $\frac{1}{2}$ & $\frac{1}{3}$ reducantur ad idem nomen, ut fit in communi Arithmetica, erunt novi exponentes $\frac{3}{6}$ & $\frac{2}{6}$, qui docent ſignum radicale commune eſſe $\sqrt[6]{}$, & quantitatem $b+\sqrt[2]{cd}$ elevandam eſſe ad tertiam poteſtatem; quantitatem vero $a-\sqrt[3]{bc}$ ad ſecundam. Fa-

ſta itaque operatione, habentur aliæ duæ radicales univerſales æquales prioribus & ejuſdem nominis, nempe

$$\sqrt[6]{b^3+3bcd+\overline{3b^2+cd}\times\sqrt{cd}}\ \&\ \sqrt[6]{a^2-2a\sqrt[3]{bc}+\sqrt[3]{b^2c^2}},$$

Radicales autem illæ, quæ ſub ſigno univerſali $\sqrt[6]{}$ comprehenduntur, hoc eſt $\sqrt[2]{cd}$, $\sqrt[3]{bc}$, & $\sqrt[3]{b^2c^2}$, reduci facile poſſunt ad idem ſignum $\sqrt[6]{}$ *per Schol. Prop.* I. *hujus*: erunt $\sqrt[2]{cd}=\sqrt[6]{c^3d^3}$, $\sqrt[3]{bc}=\sqrt[6]{b^2c^2}$, & $\sqrt[3]{b^2c^2}=\sqrt[6]{b^4c^4}$.

II. Reducenda ſit ad ſimpliciorem expreſſionem radicalis univerſalis ex. gr. $\sqrt{a^2b+a^2c\sqrt{d}}$. Dividatur a^2b+a^2c per a^2; & relicto quoto ſub ſigno univerſali, ponatur a extra ſignum, fiet $a\sqrt{b+c\sqrt{d}}$ ad terminos ſimpliciores redacta.

Similiter ſit radicalis univerſalis $\sqrt[2]{b^2c+\sqrt{b^4c^2d}}$ reducenda ad ſimpliciorem. Reducatur primo $\sqrt{b^4c^2d}$, dividendo illam per b^4c^2, fiet $b^2c\sqrt{d}$, quæ ſi reponatur ſub ſigno univerſali, erit $\sqrt[2]{b^2c+b^2c\sqrt{d}}$: hac demum ad ſimpliciorem redacta, eam ſcilicet dividendo per b^2, habebitur $b\sqrt{c+c\sqrt{d}}$, id quod quærebatur. Si fuerit $\sqrt{8+4\sqrt{3}}$, erit eadem ratione $2\sqrt{2+\sqrt{3}}$. Similiter $\sqrt{9\sqrt{12}}$, fiet $3\sqrt{2\sqrt{3}}$.

Co-

COROL. I. *Si habeatur radicalis univerſalis* $a\sqrt{b+c\sqrt{d}}$ *& ponenda ſit ſub ſigno univerſali quantitas* a, *multiplicari debet per ſecundam poteſtatem ipſius* a, *nempe per* a^2, *tota quantitas* b + c; *unde oritur* $\sqrt{a^2b+a^2c\sqrt{d}}$. *Quod ſi iterum quantitas* a^2c *ponenda eſſet ſub ſecundo ſigno, elevanda hæc eſſet pariter ad ſecundam poteſtatem, & tunc haberetur* $\sqrt{a^2b+\sqrt{a^4c^2d}}$. *Hinc luculenter apparet ratio explicatæ reductionis.*

COROLL. II. *Hoc totum convenit tam radicalibus univerſalibus, quæ habent idem ſignum radicale* $\sqrt[m]{a+\sqrt[m]{b}}$, *quam iis, quarum exponens eſt diverſus, ut* $\sqrt[m]{a+\sqrt[n]{b}}$, *habita tamen ratione eorum, quæ dicta ſint* in Prop. 2. hujus *de reductionibus.*

PROPOSITIO XII.
Radicalium univerſalium calculus.

I. UT addantur, & ſubtrahantur, reduci ſemper debent ad idem nomen & ad ſimpliciorem expreſſionem, ſi fieri poſſit, *per Propoſ. antec.* Deinde per ſigna + & — habetur additio, vel ſubtractio, ut in ceteris radicalibus. Sint addenda $3\sqrt{a+b\sqrt{c}}$ & $2\sqrt{bc+\sqrt{d}}$, ſumma erit $3\sqrt{a+b\sqrt{c}}+2\sqrt{bc+\sqrt{d}}$; reſiduum erit $3\sqrt{a+b\sqrt{c}}-2\sqrt{bc+\sqrt{d}}$.

II. Quod

II. Quod ſi radicales habeant ſub ſignis quantitates eaſdem, adduntur, vel ſubtrahuntur earum coefficientes. Sic $\sqrt{8+\sqrt{6}}$ & $\sqrt{18+\sqrt{6}}$ (quæ ſunt $2\sqrt{2+\sqrt{6}}$ & $3\sqrt{2+\sqrt{6}}$ *per Propoſ. autec.*) ſumma erit $5\sqrt{2+\sqrt{6}}$; reſiduum vero (ſubtrahendo primam ex ſecunda) erit $1\sqrt{2+\sqrt{6}}$.

III. Sit multiplicanda $\sqrt{3+\sqrt{5}}$ per $\sqrt{2+\sqrt{2}}$, deleto ſigno univerſali, multiplicantur mutuo quantitates, & facto præfigitur ſignum radicale univerſale.

Exempl. I.

$$\begin{array}{l} 3+\sqrt{5} \\ 2+\sqrt{2} \\ \hline 6+2\sqrt{5} \\ \quad +3\sqrt{2}+\sqrt{10} \\ \hline \sqrt{6+2\sqrt{5}+3\sqrt{2}+\sqrt{10}} \end{array}$$

Exempl. II.

$$\begin{array}{l} \sqrt[3]{-\frac{1}{2}q-\sqrt{\frac{1}{4}q^2-\frac{1}{27}p^3}} \\ \sqrt[3]{-\frac{1}{2}q-\sqrt{\frac{1}{4}q^2-\frac{1}{27}p^3}} \\ \hline \frac{1}{4}q^2+\frac{1}{2}q\sqrt{\frac{1}{4}q^2-\frac{1}{27}p^3} \\ \quad +\frac{1}{2}q\sqrt{\frac{1}{4}q^2-\frac{1}{27}p^3}+\frac{1}{4}q^2-\frac{1}{27}p^3 \\ \hline \sqrt[3]{\frac{1}{2}q^2-\frac{1}{27}p^3+q\sqrt{\frac{1}{4}q^2-\frac{1}{27}p^3}} \end{array}$$

IV. Si vero radix univerſalis per ſe ipſam multiplicanda ſit, & ſignum univerſale ſit potentiæ ſecundæ hoc eſt $\surd$, tunc

tunc habetur productum, deleto ſigno ipſo univerſali, quod præcedit, reliquis radicalibus, cujuſcunque gradus ſint, minime immutatis. Sic $\sqrt{\frac{5}{4}+\sqrt{101}} \times \sqrt{\frac{5}{4}+\sqrt{101}}$, deleto ſigno univerſali, dat productum $\frac{5}{4}+\sqrt{101}$. Similiter $\sqrt[2]{7+\sqrt[3]{4}+\sqrt[4]{5}} \times \sqrt[2]{7+\sqrt[3]{4}+\sqrt[4]{5}}$ dat productum $7+\sqrt[3]{4}+\sqrt[4]{5}$. Ratio eſt, quia ſumma illa terminorum $7+\sqrt[3]{4}+\sqrt[4]{5}$ ex hypotheſi erat quadratum. Deleto itaque ſigno radicali quadratico, habetur quadratum, hoc eſt quantitas illa in ſe ducta.

V. Dividenda ſit radicalis univerſalis $\sqrt{15+\sqrt{7}}$ per $\sqrt{3}$; diviſis 15 per 3, & 7 per 3, habentur quoti 5 & $\frac{7}{3}$: quibus ſi præponantur ſigna radicalia, erit quotus quæſitus $\sqrt{5+\sqrt{\frac{7}{3}}}$.

VI. Quod ſi diviſor ſit binomium, aut polynomium, ut ſi dividenda ſit $\sqrt{3+\sqrt{5}}$ per $\sqrt{2+\sqrt{3}}$, deleto ſigno univerſali, multiplicatur per quantitatem contrariam tam diviſor, quam dividendus, ut docuimus *Prop. 6. hujus.* Fiet novus diviſor $=1$, & dividendus $\sqrt{6+\sqrt{20}-\sqrt{27}-\sqrt{15}}$. Ecce exemplum

Diviſor	$2+\sqrt{3}$	$3+\sqrt{5}$
	$2-\sqrt{3}$	$2-\sqrt{3}$
	$4+2\sqrt{3}$	$6+2\sqrt{5}$
	$-2\sqrt{3}-3$	$-3\sqrt{3}-\sqrt{15}$
	$4-3=1$	$\sqrt[2]{6+2\sqrt{5}-3\sqrt{3}-\sqrt{15}}=$

Quantitas dividen. $\sqrt[2]{6+\sqrt{20}-\sqrt{27}-\sqrt{15}}$.

Schol. *Ceterum hujuſmodi calculus raro occurrit. Interea tamen tyrones in eodem exerceri non erit inutile.*

PROPOSITIO XIII.

Radicum, quæ imaginariæ dicuntur, calculum expedire.

I. Radices imaginariæ ad idem ſignum radicale & ad ſimpliciorem expreſſionem, ſi fieri id poſſit, reducuntur; tum etiam adduntur, & invicem ſubtrahuntur iiſdem omnino regulis, quas pro ceteris radicalibus tradidimus, ideoque iis non immoramur.

II. Si vero multiplicandæ ſint, & ambæ habeant ſub ſigno eandem quantitatem, obſervetur, an ſigna radicalia præferant idem ſignum +, aut —, an vero diverſum. Nam in primo caſu productum erit reale negativum, in ſecundo reale poſitivum. Ecce exempla

$$+1\sqrt{-a}\times+1\sqrt{-a}=+1\times-a=-a$$
$$-1\sqrt{-a}\times-1\sqrt{-a}=+1\times-a=-a$$
$$+1\sqrt{-a}\times-1\sqrt{-a}=-1\times-a=+a$$

Idem omnino fit, etiamſi deſit unitas, quæ ſemper intelligitur adeſſe tanquam radicalium coëfficiens, ſi nullum aliud adſit coëfficiens, ſcilicet

$$+\sqrt{-a}\times+\sqrt{-a}=+\times-a=-a$$
$$-\sqrt{-a}\times-\sqrt{-a}=+\times-a=-a$$
$$+\sqrt{-a}\times-\sqrt{-a}=-\times-a=+a$$

Ratio horum eſt, quia quod oritur ex $\sqrt{-a}\times\sqrt{-a}$, ſem-

per

per eſt $-a$, utpote quadratum negativum. Sed hoc ipſum productum negativum duci debet ulterius in ſignum illud $+$, aut $-$, quo ſigna radicalia afficiuntur, proinde ex communi lege multiplicationis ducendo $-a$ in $+$, habetur $-a$, ut in primo & ſecundo exemplo, & in utroque caſu reſtituitur quantitas realis negativa $-a$. At vero ducendo $-a$ in $-$ oritur quantitas poſitiva $+a$, ut in tertio exemplo.

III. Si vero quantitas ſub-ſigno non eſt eadem, aut ſi multiplicanda ſit radix imaginaria per realem; ut tollatur omne æquivocum, præſtat factum exprimere per ſignum multiplicationis $\times$. Multiplicanda ſit $a\sqrt{-a}$ per $-\sqrt{-b}$, productum erit $-a\sqrt{-a}\times\sqrt{-b}$, non autem $-a\sqrt{+ab}$. Per multiplicationem enim radicalium imaginariarum non oritur quantitas realis negativa, niſi cum ipſa radix imaginaria elevatur ad poteſtatem, cujus exponens ſit idem cum exponente ſigni radicalis. Sic $\sqrt{-a}\times\sqrt{-a}$ producit quantitatem realem negativam $-a$, ut ſupra explicavimus. In aliis vero caſibus habetur quantitas imaginaria; ut ſi multiplicetur $\sqrt[2]{-b}$ per $\sqrt[2]{b}$, productum, quod oritur $\sqrt[2]{-b^2}$, continet radicem imaginariam, ita ut ſi ponere velis b extra ſignum, neceſſario exprimi debeat $b\sqrt{-1}$, ut nimirum ſervetur ratio radicis imaginariæ; nec ipſa quantitas imaginaria b confundatur cum quantitatibus realibus Proinde ad evitandam confuſionem, quæ dari facile poſſet, præſtat multiplicationem peragere ſigno $\times$. Ecce exempla

$$+a\sqrt{a}\times-\sqrt{-b}=-a\sqrt{a}\times\sqrt{-b}$$
$$+2\sqrt{3}\times-1\sqrt{-5}=-2\sqrt{3}\times\sqrt{-5}$$
$$+3\sqrt{-c}\times-2=-6\sqrt{-c}$$

IV. Simili ratione expedit diviſionem per modum fractionis exprimere ad confuſionem evitandam. Dividendo $\sqrt{-a}$ per $\sqrt{-b}$, quotus erit $\frac{\sqrt{-a}}{\sqrt{-b}}$. Item dividendo $\sqrt{-3}$ per $\sqrt{-5}$, erit quotus $\frac{\sqrt{-3}}{\sqrt{-5}}$. Sic $\frac{c}{\sqrt{-a}}$ & $\frac{\sqrt{-n}}{\sqrt{m}}$ ſignificant quantitatem realem diviſam per imaginariam, & imaginariam per realem, ut patet.

V. Quod ſi eadem quantitas imaginaria tam in numeratore, quam in denominatore reperitur, ex communi lege diviſionis utrinque deletur. Sic dividendo $a\sqrt{-d}$ per $b\sqrt{-d}$, ſcribatur $\frac{a\sqrt{-d}}{b\sqrt{-d}}$, quotus erit $\frac{a}{b}$. Item ſi dividatur $-b$ per $\sqrt{-b}$, prodit quotus $\sqrt{-b}$. Nam ſi fiat $-b = \sqrt{-b} \times \sqrt{-b}$, erit $\frac{\sqrt{-b} \times \sqrt{-b}}{\sqrt{-b}} = \sqrt{-b}$, quotus quæſitus. Dividendo autem $\sqrt{-b}$ per $-b$, erit diviſio $\frac{\sqrt{-b}}{\sqrt{-b} \times \sqrt{-b}} = \frac{1}{\sqrt{-b}}$ quotus. Similiter quotus ex quantitate $a\sqrt{-bb}$ diviſa per $-bb$, erit $\frac{a\sqrt{-bb}}{\sqrt{-bb} \times \sqrt{-bb}} = \frac{a}{\sqrt{-bb}}$ Tandem dividendo $+\sqrt{-3}$ per $-\sqrt{-3}$, erit quotus -1. Nam $\frac{+1\sqrt{-3}}{-1\sqrt{-3}} = \frac{+1}{-1} = -1$.

SCHOL. *Quidam non obſcuri nominis Algebriſtæ docent productum ex multiplicatione duarum quantitatum imaginariarum, aut quotum ex diviſione illarum, eſſe quantitatem veram & realem. Sic multiplicando v. g.* $\sqrt{-3}$ *per*

$\sqrt{-2}$

$\sqrt{-2}$ *oriri* $\sqrt{6}$, *non autem* $\sqrt{-6}$. *Similiter dividendo* $\sqrt{-6}$ *per* $\sqrt{-2}$ *oriri quotum realem* $\sqrt{3}$, *non autem* $\sqrt{-3}$. *Proinde libere aſſerunt hallucinatum fuiſſe* Jac. Ozananum (*a*) *qui contrarium docuit. At quicquid illi dicant*, Ozananum *ſequitur* Cl. Wolfius (*b*) *Mathematicus noſtri temporis celebratiſſimus, qui contendit in multiplicatione quantitatum imaginariarum ſignum negativum non eſſe mutandum, ſed facto perinde ac factoribus præfigi debere ſemper ſignum* —. Alias enim, *inquit*, factores imaginarii efficerent factum reale, quod eſt abſurdum. *Quamobrem vult ſignorum regulam obſervari tantum reſpectu radicum, minime vero reſpectu quantitatum ſub ſigno radicali exiſtentium. Placet hic unum, vel alterum ipſius exemplum multiplicationis afferre.*

EXEMPL. I.

$$\begin{array}{r} \sqrt{-3}+\sqrt{-2} \\ +\sqrt{-3} \\ \hline -3+\sqrt{-6} \end{array}$$

EXEMPL. II.

$$\begin{array}{r} 3\sqrt{-5}+2\sqrt{-3} \\ 3\sqrt{-5}-2\sqrt{-2} \\ \hline -45+6\sqrt{-15} \\ -6\sqrt{-10}-4\sqrt{-6} \\ \hline -45+6\sqrt{-15}-6\sqrt{-10}-4\sqrt{-6} \end{array}$$

COROLL. *Ex allatis exemplis ſatis liquet, quantitates imaginarias complexas eodem modo multiplicari, ac cete-*

(*a*) Nouveaux Elemens liv. 1. sect. 4. pag. m. 315.

(*b*) Elementa Analy. edit. 2. cap. 2.

ras radicales complexas. Exercitationis tamen gratia nonnulla adducimus tam multiplicationis, quam diviſionis complexæ exempla, in quibus quantitates imaginariæ realibus adhærent. **A** *eſt quantitas multiplicanda,* **B** *multiplicans,* **P** *vero productum.*

EXEMPL. I.

$$\begin{array}{ll} A & \sqrt{a}-2\sqrt{-b} \\ B & \sqrt{c}+3\sqrt{-b} \\ \hline & \sqrt{ac}-2\sqrt{c}\sqrt{-b} \\ & \quad +3\sqrt{a}\sqrt{-b}-6\times-b=+6b \\ \hline P & \sqrt{ac}-2\sqrt{c}\sqrt{-b}+3\sqrt{a}\sqrt{-b}+6b \end{array}$$

EXEMPL. II.

$$\begin{array}{ll} A & 2+\sqrt{-5} \\ B & 3-\sqrt{-2} \\ \hline & 6+3\sqrt{-5} \\ & -2\sqrt{-2}-\sqrt{-2}\times\sqrt{-5} \\ \hline P & 6+3\sqrt{-5}-2\sqrt{-2}-\sqrt{-2}\times\sqrt{-5} \end{array}$$

Diviſionis exempla, in quibus *A* ſignificat dividendum, *D* diviſorem; *Q* quotum; zeri autem quantitates ex actuali diviſione deletas.

EXEMPLUM I.

$$\begin{array}{ll} A & x^2-ax-x\sqrt{-b}+x\sqrt{-c}-a\sqrt{-c}-\sqrt{-b}\times\sqrt{-c} \\ & \;\;0 \quad\;\; 0 \quad\;\; 0 \quad\;\;\; 0 \quad\;\;\; 0 \quad\;\;\; 0 \quad\;\;\; 0 \\ \hline D & x+\sqrt{-c} \\ \hline Q & x-a-\sqrt{-b} \end{array}$$

EXEM-

EXEMPLUM II.

A	$9\sqrt{-30} - 6\sqrt{-10} + 6\sqrt{-18} - 4\sqrt{-6}$
D	$3\sqrt{-5} + 2\sqrt{-3}$
Q	$3\sqrt{-6} - 2\sqrt{-2}$

CAPUT V.

De Æquationibus ſimplicibus.

DEFINITIONES.

I. ÆQuatio eſt comparatio duarum quantitatum, quæ ſigno æqualitatis junguntur, ut $x = a$, $xx + 2ax = ab$ &c.

II. Quantitates, quæ præcedunt, & quæ conſequuntur æqualitatis ſignum, ut x & a in prima æquatione, item $xx + 2ax$ & ab in ſecunda, dicuntur *membra æquationis.*

III. Æquatio dicitur *ſimplex*, ſeu *primi gradus*, ubi quantitas incognita eſt unius dimenſionis, ut $x = a + b + c$: dicitur *quadratica*, ſeu *ſecundi gradus*, ſi ad duas dimenſiones aſſurgit, ut $xx = ab$: *cubica*, ſeu *tertii gradus* cum ad tres dimenſiones, ut $x^3 = a^2 + b^2$, & generatim dicitur æquatio *quarti*, *quinti*, *ſexti gradus &c.* ſi poteſtas incognitæ ad tales dimenſiones aſcendat: $x^n = a - b$ dicitur æquatio indeterminata, quæ determinabitur, ſi fiat $n = 2$, $= 3$, $= 4$ &c.

IV. Æqua-

IV. Æquatio dicitur *affecta*, cum incognitæ poteſtas diverſos habet dimenſionis gradus, ut $x^2 + ax = ab$, $x^3 - ax^2 + bx = cd$ &c. unde dicuntur affectæ quadraticæ, affectæ cubicæ &c. Dicitur autem *pura*, cum incognitæ poteſtas eandem ſervat ubique dimenſionem, ut $ax + bx = cd$, $xx - bxx = cdd$, $x^3 = abc$ &c.

V. *Radix* æquationis eſt valor incognitæ, quæ æquationem ingreditur; ut in æquatione $x = a + b$, *radix* eſt $a + b$: nam x tanti valet, quanti aggregatum $a + b$. Similiter in æquatione $x^2 = a - c$, extracta utrinque radice quadrata, *radix* æquationis, ſeu valor incognitæ x, eſt $\sqrt{a - c}$.

VI. Si valor incognitæ fuerit poſitivus, ut $x = 3$, radix dicitur *vera*. Si negativus, ut $x = -50$, radix dicitur *negativa*, ſeu, ut Carteſius vocat, *falſa*. Utraque tamen realis eſt. Nam ſi alteri debeam aureos 50, & non habeam, revera aſſeritur, me habere -50. Quod ſi incognitæ valor exprimatur per radicem quadratam negativam, ut $x = \sqrt{-a^2}$, radix dicitur *imaginaria*, & impoſſibilis, *per Schol.* 1. *Prop.* 8. *Cap.* 3.

Axiomata.

1. Non tollitur æqualitas, ſi unus, aut quot placuerit, termini de uno æquationis membro ad alterum, mutatis ſignis, transferantur. Ut ſi $x + 2 = 5$, non tollitur æqualitas, ſi fiat $x = 5 - 2$.

2. Non tollitur æqualitas, ſi utrique æquationis membro addatur, aut ſubtrahatur quantitas æqualis. Vel ſi per eandem, aut æqualem quantitatem multiplicentur, aut dividan-

vidantur ambo æquationis membra. Nam ſit $x = 3$, multiplicando per 3, erit $3x = 9$; aut dividendo per 3, erit $\frac{1}{3}x = 1$.

3. Non tollitur æqualitas, ſi utrumque membrum elevetur ad eandem poteſtatem; vel utrinque eadem radix extrahatur. Sit enim $y = ab$, erit $y^n = \overline{ab}^n$. Contra vero ſi $y^2 = a - b$, extracta radice utrinque, erit $y = \sqrt{a - b}$.

4. Non tollitur æqualitas, ſi in alterutro æquationis membro, loco unius quantitatis, ſive complexæ, ſive incomplexæ, ponatur alia quantitas æqualis. Ut ſi $x^2 = ay + by$, & ponatur $d = a + b$, erit $x^2 = dy$. Hoc dicitur *ſubſtituere*, quod in Analyſi magnum & perpetuum uſum habet, ut deinceps videbitur.

PROPOSITIO I.

Explicantur regulæ reductionis Æquationum.

Omnis æquatio continere ſolet quantitates notas cum incognitis permixtas ſecundum propoſiti problematis conditiones. Debent itaque quantitates illæ tamdiu verſari & immutari, donec æquatio illa ſimpliciſſimam formam obtineat, quæ ſit velut ultima concluſio, ad quam tota problematis difficultas reducatur, eaque dicitur *æquatio finalis*. Hoc autem præſtant ſequentes regulæ.

*Reg.*ª 1ª Æquatio reducitur ad pauciores terminos transferendo terminos illius ex una parte in aliam ſub ſigno contrario, nam *per Ax. 1.* remanet æqualitas. Sit $x - 3 = 12$,

erit per tranſpoſitionem $x = 12 + 3$, hoc eſt $x = 15$. Similiter ſi $x - b = a$, erit $x = a + b$. Patet ratio ex 2. *Ax.*, nam additur utrinque $+ 3$, & in ſecundo exemplo additur $+ b$.

Eadem ratione ſi fuerit $x + 3 = 5$, erit per tranſpoſitionem $x = 5 - 3$, hoc eſt $x = 2$. Item $x + b = a$, erit $x = a - b$. Patet ratio ex 2. *Axiom.* Nam ſubtrahitur utrinque $- 3$, & in ſecundo exemplo ſubtrahitur $- b$. Hæc tranſpoſitio vulgo *Antitheſis* dicitur.

Reg^a^. 2^a^. Si ſint fractiones, æquatio reducitur per multiplicationem, hoc eſt multiplicando terminos omnes per denominatores fractionum, ducendo ſcilicet omnes denominatores, unum poſt alium, in totam æquationem; ſeu, quod idem eſt, ducendo ſemel productum omnium denominatorum in æquationem, & deinde fractiones ad ſimplices terminos reducendo. Sit æquatio $\frac{abd}{c} + dx = \frac{bcd}{a}$, multiplicetur primo tota æquatio per c, deinde per a, vel ſemel per ac, & habetur $\frac{aabcd}{c} + acdx = \frac{abccd}{a}$, quæ, redactis terminis *per Propoſ.* 2. *Cap.* 2., erit $aabd + acdx = bccd$.

Similiter ſit æquatio cum fractionibus, ut inferius *Probl.* 4.

$$a + \frac{x - a}{n} = \frac{2a + x - 3a - \frac{x + a}{n}}{n}$$

Multiplicetur primo per n, ſit *per Axiom.* 4. *Cap.* 2.

$$an + x - a = 2an + x - 3a - \frac{x + a}{n}$$

Rurſus

Rurſus multiplicetur per n, erit

$$an^2+nx-an=2an^2+nx-3an-x+a$$

Ac deletis ex utraque parte terminis æqualibus, & ordinata æquatione, ut in *Coroll. 1. & 3.*, quæ habentur in fine hujus, explicatur, fiet æquatio $x=an^2-2an+a$.

Regula hæc patet ex *2. Axiom.* Multiplicatur enim tota æquatio per æqualia. Nam ſublato denominatore, fractio multiplicata intelligitur *per Axiom. 4. Cap. 2.*

Regª. 3ª. Reductio fit dividendo utrumque membrum per eandem quantitatem, ut ſi fuerit $3x=12$, dividendo per 3, erit $x=4$ æquatio priori æqualis *per Axiom. 2.* Similiter ſit $ax+bx=ac$, dividendo per $a+b$, erit $x=\frac{ac}{a+b}$. Demum $axx+fxx=gb$, erit $xx=\frac{gb}{a+f}$.

Regª. 4ª. Reducitur æquatio per elevationem terminorum ad aliquam poteſtatem, cum quantitas incognita ſigno radicali implicatur. Ut ſi fuerit $\sqrt{xx-aa}+b=c$, relinquatur ex una parte æquationis quantitas ſigno radicali affecta, ita ut ſit $\sqrt{xx-aa}=c-b$: tum utraque pars æquationis elevetur ad quadratum, erit $xx-aa=c^2-2cb+b^2$. Similiter ſi fuerit æquatio $m-\sqrt{4dx+4d^2}=\sqrt{2dx+d^2}$, elevato utroque membro ad quadratum, habetur

$m^2-2m\sqrt{4dx+4d^2}+4dx+4d^2=2dx+d^2$; factaque reductione,

$$m^2-2m\sqrt{4dx+4d^2}+2dx+3d^2=0$$

Ponatur ſeorſim in altero membro quantitas radicalis, quæ remanet, erit

$$m^2 + 2dx + 3d^2 = 2m\sqrt{4dx + 4d^2}$$

& rurſus elevato utroque membro ad quadratum, evaneſcit quantitas radicalis, & habetur *per Lem. Cap.* 1.

$$m^4 - 12dxm^2 - 10d^2m^2 + 4d^2x^2 + 12d^3x + 9d^4 = 0.$$

Patet ex 3. *Ax.* Hoc vulgo dicitur, *Aſymmetriam* tollere.

Reg^a. 5^a. Reducitur æquatio, cum ex utroque ejus membro extrahitur radix; ut ſi fuerit $xx = 16$, extracta utriusque radice, erit $x = 4$. Similiter æquatio $xx = a^2 - 2ab + b^2$, extracta radice fit $x = a - b$. Demum $x^3 = 125$, radice cubica extracta, fit $x = 5$. Sequitur ex 3. *Axiom.*

Reg^a. 6^a. Reducitur æquatio, ſeu plures æquationes ad unam, ſubſtituendo valorem unius incognitæ loco ipſius. Ut ſi ſit æquatio $x - y = d$, ſit autem $a - x$ valor incognitæ y, qui ſubſtitui debet loco ipſius y: facta ſubſtitutione, habetur æquatio $x - a + x = d$, (variatis ob $-y$ ſignis in $a - x$) hoc eſt habetur *per Reg.* 1. $2x = a + d$, & *per Reg.* 3. $x = \frac{a+d}{2}$. Eadem ratione ſit æquatio $x^2 + y^2 = d^2$, & $a - x$ valor ipſius y, qui ut ſubſtituatur loco $+y^2$, fiat quadratum ex $a - x$, nempe $a^2 - 2ax + x^2$: quo poſito loco ipſius y^2, erit æquatio $x^2 + a^2 - 2ax + x^2 = d^2$; ſeu $2x^2 - 2ax = d^2 - a^2$, & $x^2 - ax = \frac{d^2 - a^2}{2}$ *per Reg.* 1., & 3.

Coroll. I. *Ex Reg.* 1. *tria ſequuntur valde utilia: primo relinquendo incognitam in una æquationis parte, & trans-*

transferendo omnes alias quantitates in partem alteram, habetur valor illius incognitæ. Sic æquatio $x+y=100$ *per tranſpoſitionem fiet* $x=100-y$, *&* $100-y$ *dicitur valor incognitæ* x.

COROLL. II. *Inſuper quantitates negativæ fieri poſſunt poſitivæ, & e contrario; transferendo ſcilicet illas in partem oppoſitam ſub ſigno contrario. Sic ut in æquatione* $a-x=b+c$, *incognita* x *fiat poſitiva, & haberi poſſit ejus valor, facta tranſpoſitione quantitatum in alteram partem ſub ſigno contrario, erit* $a-b-c=x$.

COROLL. III. *Cum in utroque membro occurrit eadem quantitas ſub eodem ſigno, deleri poteſt utrinque, & æquatio ſimplicior fit, ut ſi fuerit* $xx+ab-c=d-c+ab$, *æquatio reducitur ad* $xx=d$.

SCHOL. *In reducendis æquationibus, illud primo curandum, ut omnes incognitæ, cujuscunque ſint gradus, in uno eodemque æquationis membro conſiſtant, in altero quantitates cognitæ, id quod per regulas jam traditas obtinetur.*

PROPOSITIO II.

Plures ſimplices æquationes ad unam reducere.

I. Sint reducendæ ad unam duæ æquationes ſimplices $x+y=a$ & $x-y=b$.

Addantur ſimul, ſumma erit $2x=a+b$. Vel ſubtrahatur ſecunda ex prima, habetur $2y=a-b$. Patet ex 2. *Axiom.*

 II. Sint

II. Sint tres æquationes ad unam reducendæ

$$\text{I.}\quad x + y = a$$
$$\text{II.}\quad x + z = b$$
$$\text{III.}\quad y + z = c$$

In prima æquatione ſumatur valor incognitæ y *per Cor.* 1. *Prop. ant.* ita ut habeatur $y = a - x$, & in ſecunda valor incognitæ z, erit $z = b - x$. Deinde hi duo valores ſubſtituantur in tertia æquatione loco y & z *per Reg.* 6. habebitur $a - x + b - x = c$; & facta reductione, erit $a + b - c = 2x$ *per Reg.* 1., ſeu $x = \frac{a+b-c}{2}$ *per Reg.* 3.

III. Sint plures æquationes ad unicam reducendæ, nempe *A*, *B*, *C*, *D*.

A	$x + y + z = a$	E	$x = a - y - z$
B	$x + y + v = b$		
C	$x + z + v = c$		
D	$y + z + v = d$		
F	$a - z + v = b$	*H*	$z = a + v - b$
G	$a - y + v = c$	*M*	$y = a + v - c$

N $\quad 2a + 3v - b - c = d$

hoc eſt $v = \frac{b + c + d - 2a}{3}$

Sumatur ab æquatione *A* valor quantitatis x, qui *per Cor.* 1. *Prop. ant.* erit *E*: quo poſito in æquationibus *B* & *C*, oriuntur duæ æquationes ejusdem valoris *F* & *G*. Sumatur deinde ex æquatione *F* valor incognitæ z, oritur æquatio *H*. Similiter ex æquatione *G* ſumpto valore incognitæ y, habetur

habetur æquatio *M.* Hi duo valores ponantur in æquatione *D*, oritur æquatio *N*, in qua non eſt niſi unica incognita *v*, cujus valor $\frac{b+c+d-2a}{3}$ ſi ſubſtituatur in æquationibus *H* & *M* loco quantitatis *v*, ſtatim innoteſcunt *z* & *y*, quorum valor ſi tandem ponatur in æquatione *E*, nota erit quoque incognita *x*. Quod &c.

CoROLL. *Ut hæc reductio fieri poſſit, neceſſe eſt, ut eadem incognita in pluribus æquationibus reperiatur, ut ex allatis exemplis manifeſtum eſt.*

IV. Quod ſi non omnes incognitæ ſint unius dimenſionis, ut in æquationibus *A* & *B*, quæ ſequuntur,

$$A \quad xyy = bc, \quad B \quad xx + yy = by - ax$$

in quibus una tantum incognita *x* eſt unius dimenſionis; tunc quæritur valor hujus, & in alia æquatione ſubſtituitur hac ratione:

Quia $xyy = bc$, erit $x = \frac{bc}{yy}$ *per Reg. 3.* & $xx = \frac{b^2c^2}{y^4}$ *per Ax. 9.* Item $ax = \frac{abc}{yy}$ *per Axiom. 2.* His valoribus in æquatione *B* ſubſtitutis *per Axiom. 4.* habetur

$$\frac{b^2c^2}{y^4} + yy = by - \frac{abc}{yy}$$

Et multiplicando omnes terminos per y^4, ut fractiones exterminentur, oritur æquatio, in qua unica tantum incognita,

$$b^2c^2 + y^6 = by^5 - abcy^2$$

Sch. Prop. 1. $\quad y^6 - by^5 + abcy^2 = -b^2c^2$

PRO-

PROPOSITIO III.

Theoremata nonnulla explicantur, quæ æquationibus conficiendis inſerviunt.

Theor. 1. In proportione Arithmetica trium terminorum $a, a+1, a+2$, ſumma termini primi, & tertii æquatur duplo ſecundi, nempe $a+a+2$ (hoc eſt $2a+2$) $= 2a+2$. Item in numeris Arithmetice proportionalibus 7, 5, 3, erit $7+3=10$.

Coroll. I. *Semiſſis aggregati ex primo & tertio dat terminum ſecundum. Sic* $\frac{2a+2}{2}=a+1$. *Item* $\frac{7+3}{2}=5$.

Coroll. II. *Datis duobus mediis quatuor continue proportionalium Arithmetice, ut* a, x, *facile innoteſcit primus, quem voco* y. *Nam ſi retrocedendo fiat* $x . a :: a . y$, *erit* per Theor. $x+y=2a$, & $y=2a-x$. *Eadem ratione innoteſcit etiam quartus, quem item voco* y. *Nam ſi fiat* $a . x :: x . y$, *erit* per Theor. $a+y=2x$, & $y=2x-a$. *Sunt ergo* $2a-x, a, x, 2x-a$ *quatuor termini continue proportionales Arithmetice. Quod nota ex* Newtono (*a*).

Theor. 2. In proportione Arithmetica quatuor terminorum, ſumma duorum extremorum æquatur ſummæ duorum mediorum; ut ſi fuerint $a . a+1 :: a+2 . a+3$, erit $2a+3=2a+3$. Sic etiam in numeris Arithmetice proportionalibus $2 . 5 :: 15 . 18$, erit $2+18=15+5$.

Theor.

(*a*) Arithm. Univerſ. Probl. Geom. XIV.

Theor. 3. Quantitatum duarum inæqualium major æquatur aggregato ex ſemiſumma & ſemidifferentia earum: minor vero differentiæ inter ſemiſummam & ſemidifferentiam earundem quantitatum. Sic partium inæqualium 7 & 5, quæ componunt 12, ſemiſumma eſt 6, ſemidifferentia 1, pars major erit 6 + 1, minor 6 — 1.

COROL. *Si ponatur ſumma* = 2a *& differentia* = 2y, *erit pars major* a + y, *minor* a — y. *At ſi ſemiſumma ponatur* x *& ſemidifferentia* y, *quantitas major erit* x + y, *minor* x — y. *Quod notetur.*

Theor. 4. In proportione Geometrica quatuor terminorum, factum ex duobus extremis æquatur facto ex duobus mediis; hoc eſt ſi *a . b :: c . d*, erit *ad* = *bc*. *Eſt Propoſ.* 16. *l.* 6. *Eucl.*

COROLL. I. *Hinc ſemper quartus terminus proportionalis exprimi poteſt per factum ſecundi & tertii diviſum per primum. Sit enim* a . b :: c . d, *erit ex* hoc Theor. bc = ad *& dividendo utrumque per* a, *erit* $\frac{bc}{a}$ = d : *proinde* a . b :: c . $\frac{bc}{a}$.

COROLL. II. *Hinc quoque certo ſcire poſſumus, an quatuor termini ſint proportionales, obſervando, an productum ex duobus extremis æquale ſit producto ex duobus mediis. Nam termini proportionales varie diſponi & permutari poſſunt ſecundum varios argumentandi modos, quos in toto fere lib.* 5. *Euclides docuit. Ut ſi ſupponatur* a . b :: c . d, *erit* permutando a . c :: b . d, *&* invertendo c . a :: d . b, *&* componendo c + a . a :: d + b . b, *& ſic ulterius. Nihilominus in omnibus his permutationibus hoc Theorema ſemper verum eſſe debet.*

Theor. 5. In proportione Geometrica continua trium terminorum factum primi & tertii termini æquatur quadrato ſecundi. Nam cum ex hypotheſi $a.b :: b.c$, erit *per Theor. 4.* $ac = bb$.

Coroll. I. *Datis primo & tertio termino*, a & c, *habetur medius proportionalis* $= \sqrt{ac}$. *Nam in tribus continue proportionalibus* a, b, c, *cum ſit* $ac = bb$, *extracta ſecunda radice, erit* $\sqrt{ac} = b$. *Et generatim factum duarum quantitatum eſt medium proportionale inter quadrata illarum.*

Coroll. II. *Datis autem primo & ſecundo termino trium Geometrice proportionalium, habetur tertius, ſi fiat ex ſecundo quadratum & dividatur per primum. Nam ſit tertius* $= f$, *erunt ex hypotheſi* a, b, f *continue proportionales, ſed per hoc* Theor. 5. $af = bb$, *proinde dividendo per* a, *erit* $f = \frac{bb}{a}$. *Hac ratione inveniri poſſunt infiniti termini Geometrice proportionales.*

Coroll. III. *Denique in proportione continua Geometrica quadratum primi termini eſt ad quadratum ſecundi, ut primus terminus eſt ad tertium. Quod evidens eſt ex proportione continua* $1, a, a^2$; *tum etiam* ex Propoſ. 20. l. 6. Euclidis.

Theor. 6. Si quatuor termini Geometrice proportionales ſint, erunt quoque proportionales ad quamcunque illi dignitatem eleventur. Sit $a.b :: c.d$, erit quoque $a^2.b^2 :: c^2.d^2$. Item $a^3.b^3 :: c^3.d^3$. Demum $a^n.b^n.c^n.d^n$. Sequitur *ex Propoſ. 13. l. 8. Eucl.* vel etiam poteſt numeris illuſtrari.

Coroll.

COROLL. *Terminorum proportionalium radices ſunt pariter proportionales. Nam ſi* $a^n . b^n :: c^n . d^n$. *erit quoque* $\sqrt[n]{a} . \sqrt[n]{b} :: \sqrt[n]{c} . \sqrt[n]{d}$.

SCHOL. I. *Progreſſio Aritbmetica terminorum quorumcunque*, b, d, f, g, h *&c. exprimi ſolet per iteratam differentiæ additionem. Sit inter primum & ſecundum terminum* b *&* d *differentia* c, *erit progreſſio* b . b + c . b + 2c. b + 3c . b + 4c *&c. Quod ſi progreſſio ſit deſcendens, ita ut primus terminus* b *ſecundum ſuperet exceſſu* c, *tunc exprimitur per iteratam differentiæ ſubtractionem, nempe* b . b — c . b — 2c . b — 3c . b — 4c *&c.*

SCHOL. II. *Axiomata, reductionis regulæ & theoremata ſuperius allata ſunt fundamenta æquationum inſtituendarum; neque ſine illis reſolvi poſſunt, ut mox videbimus. Eadem minute citare non ſolent Analyſtæ, unde ſæpe tyronibus obſcuritas. Nos in gratiam eorum in duabus ſequentibus propoſitionibus id faciemus, quod deinde magna ex parte omittetur; ſiquidem memoriæ hæreant, cum eorum uſus in Analyſi perpetuus ſit.*

PROPOSITIO IV.

Quæſtionem datam ad æquationem redigere.

Deſignentur quantitates incognitæ per ultimas Alphabeti literas *x*, *z*, *v*, ut diſtinguantur a cognitis & datis, quæ literis prioribus *a*, *b*, *c*, *d* exprimi ſolent, ut diximus *Defin.* 3. ad *Cap.* 1. (quanquam & ipſæ priores literæ *a*, *b*, *c* &c. aliquando pro incognitis & indeterminatis

 uſur-

uſurpentur, ut inferius in *Problem. 6.* factum videbitur.) Expendantur deinde ſingulæ quæſtionis conditiones & relationes, quæ dantur inter cognitas & incognitas, ſeu quæſitas quantitates; & ex utriſque ſimul tot formentur æquationes, quot ſunt incognitæ aſſumptæ. Sed hæc uſu magis, quam verbis addiſcuntur. Ecce tibi plura exempla.

1. Invenire duos numeros, quorum ſumma eſt 100 & differentia 30.

Numerus major quæſitus ſit $= x$, minor $= y$, ſumma $= 100$, differentia $= 30$. Ex conditione problematis oriuntur duæ æquationes, nempe

$$x + y = 100, \ \& \ x - y = 30.$$

Vel in terminis generalibus: numerorum quæſitorum ſumma ſit $= a$, differentia $= b$, erunt duæ æquationes

$$x + y = a, \ \& \ x - y = b$$

2. Invenire duos numeros tales, ut triplum primi ſuperet ſecundum exceſſu $= a$, ſecundus vero ſuperet primum exceſſu $= b$.

Sit primus quæſitorum $= x$, alter $= y$, oriuntur duæ æquationes

$$3x = y + a, \ \& \ y = x + b$$

3. Invenire duos numeros, qui ſint in ratione 1 ad 5, ſed ſi minori addas 2 & majori 3, ſint in ratione 1 ad 2. Pone minorem $= x$, majorem $= y$, erunt duæ æquationes,

I.	$x . y :: 1 . 5$	II.	$x + 2 . y + 3 :: 1 . 2$
Theor. 4.	$y = 5x$	*Theor. 4.*	$2x + 4 = y + 3$

Pona-

Ponatur in ſecunda loco y, ejus valor ex æquatione prima inventus, hoc eſt $5x$, erit

$$\begin{array}{ll} \textit{Ax. 4.} & 2x+4=5x+3 \\ \textit{Sch. Prop. 1.} & 4-3=5x-2x \\ \textit{Reg. 3.} & 1=3x \\ & \tfrac{1}{3}=x, \ \& \ y(=5x)=\tfrac{5}{3} \end{array}$$

Adde ad $\frac{1}{3}$ numerum 2, & ad $\frac{5}{3}$ numerum 3, erit $x=\frac{7}{3}$ & $y=\frac{14}{3}$, & problema omnino reſolutum.

5. Numerum invenire, a quo ſi auferatur f, deinde g, numeri reſidui ſint in proportione m ad n.

Sit numerus quæſitus $=x$, erit per conditionem problematis

$$\begin{array}{ll} & x-f.\ x-g :: m.\ n. \\ \textit{Theor. 4.} & mx-mg=nx-nf \\ \textit{Sch. Prop. 1.} & mx-nx=mg-nf \\ \textit{Reg. 3.} & x=\dfrac{mg-nf}{m-n} \end{array}$$

Sit $f=15$, $g=20$, $m=6$, $n=1$, erit $x=\frac{105}{5}=21$. Ex quo ſi auferas 15 & 20, habebis 6 . 1 :: 6 . 1.

Vel ſit $f=3$, $g=5$, $m=2$, $n=1$, erit $x=\frac{7}{1}=7$. ex quo ſi auferas 3 & 5, habebis 4 . 2 :: 2 . 1.

6. Invenire quadratum, cui ſi addatur 6, fiat ſui lateris quintuplum.

Sit latus quadrati quæſiti x, erit per conditionem problematis

$$xx+6=5x, \ \& \ xx-5x=-6 \ \textit{per Sch. cit.}$$

7. Invenire cubum, cui ſi ter addatur ejus radix & ter auferatur quadratum radicis ejuſdem, æqualis ſit unitati.

Sit latus cubi quæſiti x, erit per conditionem problematis

$$x^3 - 3x^2 + 3x = 1.$$

Schol. I. *Quo pauciores adhibentur incognitæ, eo facilius & expeditius problema ſolvitur. Ut ſi quærantur duæ quantitates, quarum una ſit tripla alterius; ſi una denominetur* x, *præſtat alteram denominare* 3x, *quam aliam incognitam* y *inducere. Similiter ſi invenire oporteat tres continue proportionales* x, z, y. *præſtat duas tantum incognitas aſſumere, cum pro tertia ſumi poſſit* $\frac{zz}{x}$ per Coroll. 2. Theor. 5.

Schol. II. *Ceterum curent tyrones æquationes recte inſtituere. Nam ex earum efformatione facilis, aut difficilis fieri ſolet quæſtionum propoſitarum ſolutio: proinde nos multa exempla congeſſimus, plura quoque mox allaturi.*

PROBLEMA I.

Interrogatus Socrates, quænam eſſet hora, reſpondit: horæ e media nocte elapſæ ad horas usque ad meridiem reſiduas, ſe habent ut 2 ad 3: quæritur quænam eſſet hora.

Sint horæ elapſæ $= x$, erunt reſiduæ usque ad meridiem $= 12 - x$. Habetur ergo æquatio

$$x . 12 - x :: 2 . 3.$$

Theor. 4. $3x = 24 - 2x$

Reg. 1. $5x = 24$

Reg. 3. $x = \frac{24}{5}$

Erant

Erant ergo horæ $4\frac{4}{5}$, hoc eſt hora quarta matutina cum min. 48.

Vel in terminis generalibus: ſint horæ elapſæ a media nocte $= x$, horæ autem a media nocte ad meridiem $= a$, erit reſiduum ad meridiem usque $= a - x$; proportio vero horarum præteritarum ad reſiduas ſit ut n ad m, erit æquatio

$$x \,.\, a - x :: n \,.\, m.$$

Theor. 4. $$mx = na - nx$$
Sch. Prop. 1. $$mx + nx = na$$
Reg. 3. $$x = \frac{na}{m + n}$$

Sit $n = 2$, $m = 3$ & $a = 12$, erit $x = \frac{24}{5} = 4\frac{4}{5} =$ hor. 4. min. 48.

Vel ſit $n = 5$, $m = 1$, $a = 12$, erit $x = \frac{60}{6} = 10$. Fuit ergo hora decima matutina.

PROBLEMA II.

Canis leporem inſequitur, eorumque diſtantia eſt paſſuum 100; velocitas autem canis ad velocitatem leporis eſt ut 3 ad 2: quæritur ad quot paſſus canis leporem aſſequetur.

Dum canis percurrit ſpatium 100, quod voco a, lepus interea ſpatium aliquod tranſcurrit $= x$, erit ergo ſpatium a cane percurrendum $= a + x$, proinde habetur æquatio

$$a + x \,.\, x :: 3 \,.\, 2$$

Vel

Vel in terminis generalibus:

	$a + x . x :: m . n$
Theor. 4.	$mx = an + nx$
Sch. Prop. 1. .	$mx - nx = an$
Reg. 3.	$x = \frac{an}{m - n}$

Poſito igitur $a = 100$, $m = 3$, $n = 2$, erit $x = 200$, & $a + x = 300$. Aſſequetur ergo poſt paſſus 300.

SCHOL. *Hoc idem problema proponit* Fr. Lucas Pacciolus (*a*) *e Burgo Sancti Sepulcri, ſed confuſè & per ambages, quod illi* Nic. Tartalea (*b*), & Hieron. Cardanus (*c*) *duriter exprobrant.*

PROBLEMA III.

Lucilius in alenda familia quotannis impendit tritici modios 60, reliquos ſeminat. Contigit, ut ſingulis tribus annis ſextuplum ex meſſe collegerit, unde factus decuplo ditior in re frumentaria, quam antea fuerat: quæritur, quot tritici modios primo haberet.

Pone primo habuiſſe modios $= x$, modios autem 60 pro annuo familiæ alimento eſſe $= a$, meſſem ex ſemine quotannis ſextuplam $= m$; erunt modii, detractis familiæ alimentis, $= x - a$, qui ſi multiplicentur per m, dant primi anni meſſem $= mx - am$.

Secun-

(*a*) Summa de Arithm. & Geomet. proport. an. 1523. (*b*) Tom. 2. lib. I. cap. 15. quæſt. 21. (*c*) Lib. unic. de quæſt. Arithm. cap. 66.

Secundo anno detrahe rurſus a pro alimentis familiæ, erit reſiduum $mx - am - a$, & multiplicando per m, habetur ſecundi anni meſſis $m^2x - am^2 - am$.

Tertio anno, detractis ſimiliter alimentis $= a$, erit reſiduum $m^2x - am^2 - am - a$, & multiplicando per m, obtinetur fructus tertii anni $= m^3x - am^3 - am^2 - am$.

Hæc autem quantitas ex conditione problematis debet eſſe decupla modiorum, quos primo habuit, proinde erit æquatio

$$m^3x - am^3 - am^2 - am = 10x$$

Schol. Prop. 1. $$m^3x - 10x = am^3 + am^2 + am$$

Reg. 3. $$x = \frac{am^3 + am^2 + am}{m^3 - 10}$$

Subſtituantur jam pro literis, earumque poteſtatibus determinati earum valores, innoteſcet valor ipſius x. Nam poſito $a = 60$ & $m = 6$, erit $m^3 = 216$, $m^2 = 36$, $am^3 = 12960$, $am^2 = 2160$, $am = 360$; hinc $\frac{am^3 + am^2 + am}{m^3 - 10}$ $\frac{15480}{206} = 75\frac{15}{103} = x$. Habuit ergo primo modios 75 $+ \frac{15}{103}$ unius modii.

PROBLEMA IV.

Metellus duos filios teſtamento reliquit hæredes hac ratione, ut major natu accipiat aureos 100 *&* *quartam partem ejus, quod reſtat de tota hæreditate: minor vero accipiat aureos* 50 *& dimidium ejus, quod remanſit de hæreditate, detractis prius & portione ſui fratris & aureis illis* 50. *Diviſione facta, apparuit, utrumque filium eſſe ex æquo hæredem: quæritur quanta fuerit hæreditas, & quæ filiorum portio.*

Pone hæreditatem fuiſſe $= x$, aureos $100 = a$; erit reſiduum totius hæreditatis $x - a$ & ejus quarta pars $\frac{x-a}{4}$. Majoris igitur portio $= a + \frac{x-a}{4}$.

Pone ſecundo aureos $50 = b$, erit filii minoris portio $= b + \frac{x - a - b - \frac{x+a}{4}}{2}$. Sed portiones inventæ fuerunt æquales, hinc

$$a + \frac{x-a}{4} = b + \frac{x - a - b - \frac{x+a}{4}}{2}$$

Et multiplicando primo per 2, deinde per 4, erit

$$2a + \frac{2x - 2a}{4} = 2b + x - a - b - \frac{x + a}{4}$$

$$8a + 2x - 2a = 8b + 4x - 4a - 4b - x + a$$

Lem. Cap. I. $6a + 2x = 4b + 3x - 3a$

Reg. I. $x = 9a - 4b$

Cum ſit $a = 100$, $b = 50$, erit hæreditas x $(= 9a - 4b)$ $= 700$, & valore $(9a - 4b)$ in ſuperiori æquatione poſito, innoteſcit utriusque filii portio $= 250$.

Idem problema proponi poterat paulo univerſalius, quod ad tyronum exercitationem facimus.

Metellus filios ſuos reliquit hæredes hac lege, ut primus omnium A *accipiat aureos* a, *& de reliqua hæreditate partem* n. *Secundus filius* B *accipiat aureos* 2a *& partem itidem* n *de reliqua hæreditate. Filius* C *habeat aureos* 3a, *& de eo, quod remanet, partem ſimiliter* n, *& ſic de aliis. Facta inter filios diviſione, repertum eſt ſingulos eſſe æqualiter hæredes; quæritur quanta fuerit hæreditas, qui filiorum numerus, eorumque portio.*

Sit hæreditas $= x$, & accipiat A de illa portionem $= a$ reſiduum erit $x - a$, cujus pars $\frac{x - a}{n}$ ad ipſum A ſpectans.

Erit ergo filii A portio $= a + \frac{x - a}{n}$

Filii vero B portio $=2a+\frac{x-3a-\frac{x+a}{n}}{n}$, quæ portiones ex hypotheſi ſunt æquales, proinde

$$a+\frac{x-a}{n}=2a+\frac{x-3a-\frac{x+a}{n}}{n}$$

hoc eſt multiplicando per n, erit

$$an+x-a=2an+x-3a-\frac{x+a}{n}$$

& rurſus multiplicando per n totam æquationem, & auferendo terminos, qui ſuperfluunt, habetur, ordinata æquatione, valor ipſius $x=an^2-2an+a$, valor ſcilicet totius hæreditatis. Quo valore in prima æquatione $a+\frac{x-a}{n}$ ſubſtituto, invenitur portio filii $A=a+\frac{n^2a-2na}{n}=na-a$, per quam (cum portiones ſint æquales) ſi dividatur tota hæreditas jam inventa $n^2a-2na+a$, quotus $n-1$ dat numerum filiorum.

Sit $a=100$, & $n=6$, erit $an-a=600-100=500$ portio ſingulorum. At vero $n-1=6-1=5$ filiorum numerus, adeoque 5×500 dat integram hæreditatem 2500.

Coroll. *Pro generali problematis ſolutione habetur triplex Canon.* 1°. *Denominator fractionis unitate multatus* ($n-1$) *dat numerum filiorum.* 2°. *Denominator fractionis unitate multatus in quantitatem datam ductus* $\overline{n-1}\times a$ *dat filiorum*

filiorum portionem. 3°. *Denominatoris unitate multati quadratum in quantitatem datam ductum* $\overline{n-1} \times \overline{n-1} \times a$ *dat integram hæreditatem.*

PROBLEMA V.

Tres mercatores, inita ſocietate, lucrati ſunt ſummam quandam aureorum. Hoc unum conſtat, primum cum ſecundo accepiſſe ſummam aureorum a, *& eundem primum cum tertio aureos* b, *ſecundum vero cum tertio aureos* c: *quæritur, quantum fuerit lucrum ſingulorum.*

Sint tres ſocii x, z, y, ſummæ vero a, b, c; erit per conditionem Probl.

$$\begin{array}{l|l} x+z=a & z=a-x \\ x+y=b & y=b-x \\ z+y=c & \end{array}$$

Fiat reductio, ut in *Prop.* 2. docuimus, & quantitatis x valor $(\frac{a+b-c}{2})$ ponatur in duabus æquationibus $z=a-x$, & $y=b-x$, innoteſcet lucrum ſingulorum, nempe

$$x=\frac{a+b-c}{2},\ z=\frac{a-b+c}{2},\ y=\frac{b-a+c}{2}$$

Sit $a=100$, $b=120$, $c=160$, erit $x=30$, $z=70$, $y=90$.

Coroll. *Patet ſingulas alternas ſummas exced debere ſummam quamlibet tertiam, alias valor prodi negativus. Sic in prima æquatione* $a + b > c$, *in ſecun* $a + c > b$ *&c.*

Idem problema aliter. Sint tres æquationes, ut prius,

$$x + z = a, \quad x + y = b, \quad z + y = c$$

Addantur ſimul *per Prop. 2. hujus*, erit nova æquatio

$$2x + 2z + 2y = a + b + c$$

Divid. per 2. $\quad x + z + y = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$

Ex hac æquatione ſubtrahe *per Prop. cit.* tres æquationes priores ſigillatim, innoteſcet valor x, z, y.

$$\begin{array}{rl} & x + z + y = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ \textit{Subtrah.} & z + y = c \\ \hline & x = \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c \end{array}$$

$$\begin{array}{rl} & x + z + y = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ \textit{Subtrah.} & x + y = b \\ \hline & z = \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}b \end{array}$$

$$\begin{array}{rl} & x + z + y = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ \textit{Subtrah.} & x + z = a \\ \hline & y = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}a \end{array}$$

Poſito, ut ſupra, $a = 100$, $b = 120$, $c = 160$, iterum prodit $x = 30$, $z = 70$, $y = 90$.

PROBLEMA VI.

Lucius ad ædificandam domum lapides, calcem & ſabulum comparaverat. Mutato deinde conſilio ea omnia tribus emptoribus A, B *&* C *vendidit, nempe*

A emit	2. currus lapidum 3. currus calcis 7. currus ſabuli	}	*Juliis* 34
B emit	3. currus lapidum 4. currus calcis 12. currus ſabuli	}	*Juliis* 46
C emit	4. currus lapidum 1. currum calcis 13. currus ſabuli	}	*Juliis* 42

Quæritur ſingularum ejuſmodi rerum pretium.

Pone pretium lapidum pro ſingulis curribus $= a$, pro ſingulis curribus calcis $= b$, & pro curribus ſabuli $= c$; erunt ergo 2 currus lapidum $= 2a$; currus 3 calcis $= 3b$ & 7 currus ſabuli $= 7c$, (quantitates a, b, c hic ſumuntur pro incognitis & indeterminatis, ut monuimus in *Propoſ.* 4. *hujus*) habentur ergo æquationes

$$\begin{array}{ll} A & 2a + 3b + 7c = 34 \\ B & 3a + 4b + 12c = 46 \\ C & 4a + 1b + 13c = 42 \end{array}$$

Ex prima æquatione A habetur *per Schol. Prop. 1.*

$$2a = -3b - 7c + 34$$

Reg. 3. $a = -\frac{3}{2}b - \frac{7}{2}c + 17$

Hoc valore in ſecunda æquatione B poſito per *Ax. 4. hujus* erit

$$-\tfrac{9}{2}b - \tfrac{21}{2}c + 51 + 4b + 12c = 46$$

Lem. Cap. 1. $-\frac{1}{2}b + \frac{3}{2}c + 51 = 46$

Subtrab. $51 \quad 51$

$$-\tfrac{1}{2}b + \tfrac{3}{2}c = -5$$

Reg. 1. $-\frac{1}{2}b = -\frac{3}{2}c - 5$

Reg. 2. $-b = -3c - 10$

Reg. 1. $b = 3c + 10$

Adde $+\frac{1}{2}b = \frac{3}{2}c + 5$

$$\tfrac{3}{2}b = \tfrac{9}{2}c + 15$$

Sed inventa fuit $a = -\frac{3}{2}b - \frac{7}{2}c + 17$, proinde ſi in hac æquatione pro $-\frac{3}{2}b$ ponatur ejus valor $(\frac{9}{2}c + 15)$, erit *per 4. Ax. hujus.*

$$a = -\tfrac{9}{2}c - 15 - \tfrac{7}{2}c + 17$$

Lem. cit. $a = -8c + 2$

Subſtituantur jam in tertia æquatione C valores inventi a & b, obtinebitur æquatio

$$-32c + 8 + 3c + 10 + 13c = 42$$

Lem. cit. $-16c = 24$

Reg. 3. $c = -\frac{24}{16} = -\frac{3}{2}$

Sunt autem $a = -8c + 2$, & $b = 3c + 10$. Subſtituto igitur valore ejusdem c, determinantur a & b, ſcilicet

$$a = 12 + 2 = 14$$
$$b = -\tfrac{9}{2} + 10 = 5\tfrac{1}{2}$$

Proin-

Proinde lapides pro ſingulis curribus venduntur juliis 14, calcis currus quilibet juliis $5\frac{1}{2}$. Currus autem ſabuli ob valorem negativum ($c = -1\frac{1}{2}$) nihil venditur; imo ipſe ſabuli dominus ſolvit julium $1\frac{1}{2}$ emptoribus illis pro unoquoque ſabuli curru e domo amovendo.

SCHOL. I. *Hoc Problem. a* Hieronymo Cardano (*a*) *paucis verbis propoſitum, viſum eſt nobis non inelegans & tyronibus in calculo exercendis opportunum.*

SCHOL. II. *Poterant hæc problemata abſtracte, & in terminis generalibus proponi, ut plerique faciunt. Sed quia ſpecialia & particularibus circumſtantiis deducta phantaſiam magis juvant, & juvenes cum ſtatim ab initio artis, alioquin difficilis, uſum aliquem ſpecialem agnoſcunt, ad ulteriora alacriores fiunt, ea ſic afferre ſatius duximus.*

SCHOL. III. *Si ex problematis propoſiti conditionibus haberi non poſſint tot æquationes, quot incognitæ fuerunt aſſumptæ, problema dicitur* indeterminatum, *quia non unam, ſed plures obtinere poteſt ſolutiones, & tunc una ex incognitis ad arbitrium ſumitur. Ut ſi quærantur duo numeri, quorum factum æquale ſit numero dato, v.g.* 12. *Sint illi* x & y, *per conditionem problematis una tantum haberi poteſt æquatio, nempe* $xy = 12$; *unde* $y = \frac{12}{x}$. *Sumatur ad arbitrium* $x = 2$, *erit* $y = \frac{12}{2} = 6$. *Si ſumatur* $x = 3$, *erit* $y = \frac{12}{3} = 4$. *Si* $x = 4$, *erit* $y = 3$ *&c. Hujus generis ſunt problemata, quæ ſequuntur.*

(*a*) Lib. unic. de quæſt. Arithm. cap. 66.

Problemata Indeterminata.

PROBLEMA I.

Duos numeros invenire, quorum ſumma ſit ad eorum differentiam in quacunque data ratione.

Sit quæſitorum unus $= x$, alter $= y$, & ratio data ſit a ad b, erit per conditionem Probl.

$$x+y \,.\, x-y :: a \,.\, b$$

Theor. 4. $\quad ax - ay = bx + by$

Schol. Prop. 1. $\quad ax - bx = ay + by$

Divid. per $a - b \quad x = \dfrac{ay + by}{a - b}$

Sit $a = 3$, $b = 1$, & ſumatur ad arbitrium $y = 2$, erit $x = \dfrac{6+2}{2} = 4$, proinde $4 + 2 \,.\, 4 - 2 :: 3 \,.\, 1$.

Sit $a = 4$, $b = 3$, & ſumatur, ſi libet, $y = 5$, erit $x = \dfrac{20+15}{1} = 35$. Hinc $35 + 5 \,.\, 35 - 5 :: 4 \,.\, 3$

PROBLEMA II.

Duos numeros invenire, quorum ſumma ſit ad ſummam quadratorum, quæ ex ipſis fiunt, in ratione data.

Sit unus numerus quæſitus $= x$, alter $= xy$, & ratio data ſit a ad b, erit ex conditione problematis

$x +$

$$x + xy \,.\, xx + xxyy :: a \,.\, b$$

Theor. 4. $bx + bxy = axx + axxyy$

Reg. 3. $b + by = ax + axyy$

Reg. 3. $x = \frac{b + by}{a + ayy}$

Sit $a = 3$, $b = 1$, & ſumatur ad libitum $y = 2$, erit $x = \frac{1+2}{3+12} = \frac{1}{5}$, proinde $x + xy = \frac{3}{5}$, & $xx + xxyy = \frac{5}{25} = \frac{1}{5}$, ideoque $\frac{3}{5} \,.\, \frac{1}{5} :: 3 \,.\, 1$.

Sit $a = 2$, $b = 1$, & ſumatur $y = 3$, erit $x = \frac{1+3}{2+18} = \frac{4}{20} = \frac{1}{5}$, $x + xy = \frac{4}{5}$, & $xx + xxyy = \frac{10}{25} = \frac{2}{5}$, proinde habetur $\frac{4}{5} \,.\, \frac{2}{5} :: 2 \,.\, 1$.

SCHOL. I. *Eodem modo reſolvitur problema, ſi quærantur duo numeri, quorum ſumma ſit ad differentiam quadratorum ex ipſis in quacunque ratione data* a *ad* b. *Vel duo numeri, quorum differentia ſit ad differentiam quadratorum, quæ ex ipſis fiunt, in ratione data* a *ad* b. *Vel duo numeri, quorum differentia ſit ad ſummam quadratorum ex ipſis, ut* a *ad* b. *Quos omnes caſus* Diophantus Alexandrinus (*a*) *accurate reſolvit.*

SCHOL. II. *Nota pro altero numero quæſitorum non ſumi ſimpliciter* x, *vel* y, *ſed* xy, *ut ſcilicet tota æquatio multiplicata per* x *exiſtat, & inde haberi poſſit diviſor* com-

(*a*) Arithmetic. lib. 1. & 2. cum Bacheto, & de Fermat.

communis, & facta diviſione, quadrata illa ad quantitatem unius dimenſionis reducantur; quod quidem pro ſimilibus caſibus eſt advertendum.

PROBLEMA III.

Datum numerum quadratum in alia duo quadrata dividere.

Sit latus dati quadrati $=a$, latus quæſiti $=x$, & alterius quæſiti $=y$, erit $a^2=x^2+y^2$. Quia vero latus a majus eſt alterutro latere x, vel y ſeorſim accepto, fieri poterit $a-x=y$, vel ſumpto ad libitum v, fieri poterit $a-vx=y$, vel etiam $vx-a=y$; tunc enim ſuperior æquatio $a^2=x^2+y^2$ in ſequentem transformabitur, ſubſtituto, loco y^2, ejus valore.

$$a^2=x^2+v^2x^2-2avx+a^2$$

$$v^2x+x=2av$$

Divid. per v^2+1 $\qquad x=\dfrac{2av}{v^2+1}$

Hoc valore ipſius x poſito in æquatione $vx-a=y$, habetur $y=\dfrac{avv-a}{v^2+1}$.

Sit $a=10$, aſſumpto ad libitum $v=3$, erit $x=\frac{60}{10}=6$, & $y=\frac{80}{10}=8$. Fiant quadrata, erit $100=36+64$.

Sit $a=12$, & ad libitum $v=2$, erit $x=\frac{48}{5}$, & $y=\frac{36}{5}$. Fiant quadrata reducendo 12 ad fractionem ejusdem nominis, hoc eſt ad $\frac{60}{5}$, erit (deleto communi denominatore 25) $3600=2304+1296$.

PRO-

PROBLEMA IV.

Datum numerum in duos ſecare, quorum factum ſit quadratum.

Sit numerus datus $= 2a$, partium vero, in quas dividi debet, differentia ſit $= 2y$, erit *per Theorem. 3.* pars major $a+y$, minor autem $a-y$, earumque factum $aa-yy$, quod ex conditione problematis debet eſſe quadratum. Pro cujus latere, ſumpto v ad libitum, fiat $a = vy$, ſeu $vy-a$, erit

$$aa - yy = v^2 y^2 - 2avy + a^2$$

Factaque reductione *per Lem. Cap. 1.* habetur

$$v^2 y^2 + y^2 = 2avy$$

Divid. per y $\quad v^2 y + y = 2av$

Tum per $v^2+1 \quad y = \frac{2av}{v^2+1}$

Sit $2a = 20$, ſumatur ad arbitrium $v = 3$, erit $y = \frac{60}{10} = 6$, proinde $a + y = 16$, & $a - y = 4$, unde $aa - yy = 100 - 36 = 64 = 4 \times 16$.

Sit $2a = 30$, ſumpto $v = 2$, erit $y = \frac{60}{5} = 12$, proinde $a + y = 27$, $a - y = 3$, & $aa - yy = 81$.

SCHOL. *Quantitas* u *eſt quidem arbitraria, ſed talis ſumi debet, ut* y *ſemper maneat minor quam* a, *cujus differentia eſſe ſupponitur.*

PROBLEMA V.

Quadratum invenire, quod ſive addito, ſive dempto quocunque ſuæ radicis multiplici, ſubmultiplici, ſuperparticulari, & quavis alia proportionis ſpecie, ſemper ſit quadratum.

Sit quæſiti quadrati latus $=x$, & radicis addendæ, aut detrahendæ multiplum $=m$, erit x^2+mx quadratum una cum multiplo ſuæ radicis, cui quadratum æquale quæritur. Pro cujus latere, ſumpto ad libitum v, fiat $v-x$, erit ex conditione Problem.

$$x^2+mx=vv-2vx+xx$$

Sch. Prop. 1. $$2vx+mx=vv$$

Reg. 3. $$x=\frac{vv}{2v+m}$$

Sit $m=2$, & ſumatur ad libitum $v=5$, erit $x=$, hinc $x^2+mx=\frac{625}{144}+\frac{50}{12}=\frac{1225}{144}$, cujus $\surd=\frac{35}{12}$.

Sit $m=3$, & ad libitum $v=4$, erit $x=\frac{16}{11}$, & $x^2+mx=\frac{256}{121}+\frac{48}{11}=\frac{784}{121}$, cujus $\surd=\frac{28}{11}$.

Quod ſi data radicis proportio non addi, ſed ſubtrahi debeat a ſuo quadrato, tunc invenietur $x=\frac{v^2}{2v-m}$, ut calculum ineunti palam eſt.

Sit enim, ut ſupra, $m=3$, $v=4$, erit $x=\frac{16}{5}$, & $x^2-mx=\frac{256}{25}-\frac{240}{25}=\frac{16}{25}$, cujus $\surd=\frac{4}{5}$.

Sit demum $m=4$, $v=5$, erit $x=\frac{25}{6}$ & $x^2-mx=\frac{625}{36}-\frac{100}{6}=\frac{25}{36}$, cujus $\surd=\frac{5}{6}$.

Schol.

SCHOL. *In hoc problema ſatis elegans caſu incidit, ut ipſe aſſerit.* P. Auguſtinus Thomas a S. Joſepho (a) *Schol. Piarum Geometra & Aſtronomus in Germania clariſſimus. Pro quo, ſuppreſſa analyſi, canonem tradidit univerſalem* $\frac{a^4+2a^2m^2}{4a^3-4am^2}+m^4=$ *radici quadrati quæſiti; ubi* m *denotat proportionem cujuscunque ſpeciei;* a *ſumitur ad libitum, modo major ſit quam* m. *At canon a nobis traditus longe ſimplicior eſt, ut patet.*

PROBLEMA VI.

Tres numeros invenire, quorum tum ſumma, tum bini quadratum efficiant.

Sint numeri quæſiti x, y, z; primi autem quadrati $x+y+z$ latus ſit r, ſecundi quadrati $x+y$ latus ſit s, tertii $x+z$ ſit t, & quarti $y+z$ ſit v, erunt quatuor æquationes

1.ª	$x+y+z=rr$
2.ª	$x+y=ss$
3.ª	$x+z=tt$
4.ª	$y+z=vv$

Ex prima & ſecunda æquatione habentur duo valores ipſius x, *per Coroll. 1. Prop. 1.* nempe

$$x=rr-y-z,\ \&\ x=ss-y$$

$$rr-y-z=ss-y$$

Axiom. 2. $rr-z=ss$

Coroll. cit. $z=rr-ss$

Ex

(a) Sylloge Epiſt. Mathem. Pragæ an. 1713.

Ex tertia æquatione habetur $x = tt - z$ *per Coroll. cit.*, poſitoque loco z ejus valore modo invento, habetur *per Axiom. 4.*

$$x = tt - rr + ss$$

Cumque habeatur ex ſecunda æquatione $x = ss - y$, duæ æquationes æquales erunt, ſcilicet

$$tt - rr + ss = ss - y$$

Ax. 2. $tt - rr = -y$

Ax. 1. & Cor. 2. Prop. 1. $y = rr - tt$

Poſitis autem in quarta æquatione $y + z = vv$, valoribus y & z jam inventis, erit *per Ax. 4.*

$$rr - tt + rr - ss = vv$$

$$2rr - tt - ss = vv$$

Fiat $r - m = t$, erit $rr - 2rm + m^2 = tt$. Item fiat $r - n = s$, erit $rr - 2rn + n^2 = ss$; & his valoribus in ſuperiori æquatione ſubſtitutis loco $-tt - ss$, oritur æquatio

$$2rr - rr + 2rm - m^2 - rr + 2rn - n^2 = vv$$

Lem. Cap. 1. $2rm + 2rn - m^2 - n^2 = vv$

Ax. 1. $2rm + 2rn = vv + m^2 + n^2$

Reg. 3. $r = \dfrac{vv + m^2 + n^2}{2m + 2n}$

m, n, v ſunt ad libitum.

Sit $m = 1$, $n = 2$, $v = 7$, erit $r = \frac{54}{6} = 9$; hinc $r - m = t = 9 - 1 = 8$, & $r - n = s = 9 - 2 = 7$.

Proinde

	$x = 32$	$x + y + z = 81$
	$y = 17$	$x + y = 49$
	$z = 32$	$x + z = 64$
		$y + z = 49$

Sit

Sit $m=2$, $n=4$, $v=8$, erit $r=\frac{24}{12}=7$, & $r-m=t=5$, $r-n=s=3$; adeoque

$$\begin{array}{l|r} x=-15 & x+y+z=49 \\ y=24 & x+y=9 \\ z=40 & x+z=25 \\ & y+z=64 \end{array}$$

SCHOL. *Problemata indeterminata qui plura cupit, Diophantum adeat, qui in numeris infinita fere proponit, unde* Analyſis Diophantea *nomen accepit. Horum ſolutionem recentiores alia via per analyſim ſpecioſam indagarunt,* Billy (a) *præſertim*, Ozanam (b), *&* Præſtetus (c). *Hæc quidem determinatis difficiliora tyroni evadunt. Negligenda tamen non ſunt, cum in doctrina curvarum uſum habeant plurimum, ut nos opportune* Wolfius (d) *admonet. Quæ nos hic pauca ſelegimus, ad ceterorum intelligentiam, ſiquidem probe percepta fuerint, ſatis erunt.*

CAPUT VI.

De Æquationibus compoſitis.

Hactenus de æquationibus *ſimplicibus*. Nunc ad *compoſitas*, quæ pluribus conſtant dimenſionum gradibus, tranſitum facimus. Quarum naturam, & proprietates præcipuas hoc loco breviter explicabimus.

S PRO-

(a) Diophant. redivivus. (b) Nouveaux Elem. d'Algebr. l. 3. (c) Nouveaux Elemens de Mathematiq. T. 2. (d) Elem. analy. edit. 2. Cap. 2. §. 249.

PROPOSITIO I.

Explicatur æquationum compoſitarum geneſis.

Aſſumantur nonnullæ æquationes ſimplices, quæ radices poſitivas, vel negativas contineant, & æquentur nihilo. Deinde ad invicem multiplicentur, orientur æquationes compoſitæ tot graduum, quot aſſumptæ fuerint radices.

$$\text{Sit}\quad x = a,\ \text{ſeu}\ x - a = 0$$
$$x = b \qquad x - b = 0$$

Multiplicentur inter ſe hæ duæ æquationes ſimplices, oritur

$$\begin{aligned} x^2 - ax &+ ab = 0 \\ - bx & \end{aligned}$$

Sit deinde $x = c$, ſeu $x - c = 0$, & per hanc multiplicetur æquatio jam inventa, erit æquatio compoſita tertii gradus

$$\begin{aligned} x^3 - ax^2 &+ abx \\ - bx^2 &+ acx - abc = 0 \\ - cx^2 &+ bcx \end{aligned}$$

Eadem ratione ſit $x = 2$, ſeu $x - 2 = 0$
$x = -3$, ſeu $x + 3 = 0$
$x = 4$, ſeu $x - 4 = 0$

Multiplicentur ad invicem hæ tres æquationes ſimplices, fit æquatio tertii gradus

$$x^3 - 3x^2 - 10x + 24 = 0$$

Quæ

Quæ quidem ſi ulterius multiplicetur per aliam æquationem ſimplicem $x + 1 = 0$, fit æquatio quarti gradus, nempe

$$x - 2x^3 - 13x^2 + 14x + 24 = 0$$

Atque hic nonnulla circa æquationes hujusmodi diligenter obſervanda ſunt:

I. In qualibet æquatione tot dari radices, quot incognita primi termini dimenſiones habet, ſeu quot habet exponens unitates; nempe duas in quadratica, tres in cubica &c.

II. Quantitatem cognitam ſecundi termini continere ſummam omnium radicum ſub ſigno contrario, hoc eſt radices poſitivas cum ſigno —, negativas cum ſigno +. Quantitatem cognitam tertii termini exhibere productum ex ſingulis binis radicibus ſub ſigno proprio: cognitam quarti productum ex ſingulis ternis radicibus ſub ſigno contrario, & ſic deinceps. Terminum vero ultimum (quem homogeneum comparationis vocant) eſſe factum omnium radicum. Hæc omnia in prima & ſecunda æquatione oculis patent.

III. In omni æquatione tot dari radices poſitivas, ſeu veras, quot ſunt mutationes ſignorum de + in —, & de — in +; tot vero negativas, ſeu falſas, quot ſucceſſiones ſignorum eorundem + +, vel — —. Sic in æquatione $x^2 - 5x + 6 = 0$, quia duplex eſt mutatio ſignorum + —, & — +, duplex eſt radix poſitiva, nempe $x = 2$, & $x = 3$. In æquatione vero $x^2 + 3x - 4 = 0$, quia una eſt ſignorum eorundem ſucceſſio + +, & una mutatio + —, una radix eſt negativa, nempe $x = -4$,

& altera poſitiva $x=1$. Si vero omnia ſigna ſint poſitiva, ut in æquatione $x^2+5x+6=0$, radices omnes falſæ ſunt, nempe $x=-2$, & $x=-3$.

IV. Si radices veræ ſint falſis æquales, ſecundus terminus æquationis evaneſcit, & fit æqualis zero. Sit $x=2$, $x=3$, & $x=-5$. Fiat æquatio, ut ſupra docuimus, oritur $x^3 * -19x+30=0$, in qua ſecundus terminus deficit. Si radices veræ ſuperant falſas, ſecundus terminus æquationis eſt cum ſigno $-$. Sit radix vera $x-5$ major quam falſa $x=-2$, erit æquatio $x^2-3x-10=0$, in qua -3 denotat exceſſum radicis veræ $+5$ ſupra falſam -2. Si denique radices falſæ majores ſunt, quam veræ ſecundus terminus æquationis eſt cum ſigno $+$. Sit radix falſa $x=-5$, vera autem ſit $x=2$, erit æquatio $x^2+3x-10=0$, in qua $+3$ exprimit exceſſum radicis falſæ -5, ſupra veram $+2$.

V. Duplici ratione cognoſcitur, quantitatem aliquam poſitivam, aut negativam eſſe radicem æquationis. Primo ſi fiat binomium conſtans incognita, & quantitate aliqua data cum ſigno $-$, ſi ſit quantitas poſitiva; vel cum ſigno $+$, ſi ſit negativa, & per ipſum exacte, & ſine ullo reſiduo æquatio ſit diviſibilis. Sic quia æquatio ſuperior, $x^3-3x^2-10x+24=0$ diviſibilis eſt exacte per $x-2$, ſeu per $x+3$, vel per $x-4$, deducitur $+2$, -3, $+4$ eſſe radices ejusdem æquationis: cum æquationes oriantur ex multiplicatione, & diviſio ſit multiplicationi contraria. Secundo ſi ſubſtituendo in æquatione, loco incognitæ, quantitatem datam cum ſigno $+$ ſi eſt poſitiva, aut cum ſigno $-$ ſi eſt negativa, & omnia producta per ſigna contraria ſeſe deſtruant, erit illa quantitas

titas radix æquationis. Sit eadem æquatio $x^3 - 3x^2 - 10x + 24 = 0$. Subſtituatur loco incognitæ x, ubique ejus valor $+2$; erit $8 - 12 - 20 + 24 = 0$. Subſtituatur rurſus loco ejusdem incognitæ x valor -3, erit $-27 - 27 + 30 + 24 = 0$. Demum ſubſtituto valore $+4$, erit $64 - 48 - 40 + 24 = 0$.

Similiter in æquatione $x^2 \begin{matrix} -ax \\ +bx \end{matrix} - ab = 0$. Radices ſunt $+a, -b$, ſubſtituantur hi valores loco incognitæ x, erit $a^2 - a^2 + ab - ab = 0$. Item $b^2 - b^2 + ab - ab = 0$.

Schol. I. *Præter radices rationales poſitivas, aut negativas, de quibus hactenus locuti ſumus, dantur etiam in æquationibus radices irrationales, & incommenſurabiles tam poſitivæ, quam negativæ. Sic æquatio* $x^2 - 6x + 4 = 0$ *habet duas radices irrationales* $3 + \sqrt{5}$, *&* $3 - \sqrt{5}$.

Schol. II. *Præter radices reales rationales, & irrationales jam allatas, æquatio aliquando continet radices imaginarias & impoſſibiles. Nam cum nonnullæ quæſtiones caſus impoſſibiles involvant, ut ſi quæratur in circulo dato applicari rectam, quæ diametro circuli major ſit, neceſſe eſt radix illi caſui reſpondeat impoſſibilis. Sic æquatio* $x^2 - 2x + 7 = 0$ *continet duas radices imaginarias* $1 + \sqrt{-6}$ *&* $1 - \sqrt{-6}$, *quæ in æquationibus ſunt ſemper numero pares.*

Schol. III. *Regula, quam ex Carteſio docuimus ſupra* num. 3. *pro dignoſcendis ex permutatione, aut ſucceſſione ſignorum radicibus veris ac falſis, non valet pro æquationibus, quæ radicibus imaginariis conſtant. Nam ſuperior æquatio* $x^2 - 2x + 7 = 0$ *ex ſignorum permutatione con-*

tinere videtur duas radices veras, quod tamen eſt falſum. Multiplicetur enim per $x + 3 = 0$, *ut præter duas veras, aliam quoque falſam contineat: oritur* $x^3 + 1x^2 + 1x + 21 = 0$, *in qua diſpoſitio ſignorum ex præcit. regula indicat, omnes radices eſſe falſas. Non erant ergo in æquatione duæ radices veræ, quales apparebant, ſed imaginariæ* $1 + \sqrt{-6}$, *&* $1 - \sqrt{-6}$. *Proinde hic regula fallit.*

PROPOSITIO II.

Æquationem quamcunque ordinare.

I. Ponantur in uno æquationis membro omnes incognitæ, ita ut primo loco ſtatuatur incognita maximæ poteſtatis, quæ dicitur *primus* æquationis terminus: ſecundo loco incognita poteſtatis uno gradu inferioris, quæ dicitur *ſecundus* terminus æquationis, deinde omnes aliæ incognitæ gradatim decreſcentes, quæ conſtituunt *tertium*, *quartum*, *quintum &c.* terminum æquationis. In altero æquationis membro ponantur omnes termini, qui ex cognitis quantitatibus componuntur.

Sit æquatio inventa $bx^2 + x^3 = bx - ab + cd$
erit *ordinata* $x^3 + bx^2 - bx = cd - ab$

Item ſit æquatio $by^3 - dy + y^4 = abc + cy + mf$
erit *ordinata* $y^4 + by^3 - cy - dy = abc + mf$

II. Si terminus maximæ poteſtatis multiplicatus exiſtat per aliam quantitatem, ſive literariam, ſive numericam, dividi debet per illam, ut ſimplex evadat. Sit $3x^2 + 6ax = ab$, dividendo per 3 totam æquationem, erit

x^2

$x^2 + 2ax = \frac{1}{3}ab$. Eadem ratione ſi ſit æquatio $ax^2 - 2ax = abc$, dividendo per a, fit $x^2 - 2x = bc$.

III. Si terminus maximæ poteſtatis afficitur ſigno —, fieri debet per terminorum transpoſitionem poſitivus. Sic in æquatione $ax - x^2 = ab - cf$, facta terminorum transpoſitione, erit $x^2 - ax = cf - ab$.

IV. Omnes illi termini, in quibus incognita eandem habet dimenſionem, ſtatuantur unus infra alium; idemque fiat de quantitatibus cognitis, ſi plures ſint, ut

$$\begin{matrix} x^3 + ax^2 - bx \\ \quad - bx^2 + cx \end{matrix} = abc$$

Quantitates $ax^2 - bx^2$ ſtant ambæ loco ſecundi termini, & $-bx + cx$ loco tertii termini. Similiter in æquatione

$$\begin{matrix} x^2 - ax = ac - bc + mf \\ - bx \qquad\qquad\qquad \end{matrix}$$

Quantitates $ac - bc + mf$ unicum terminum conſtituunt.

COROLL. I. *Hinc habetur modus revocandi quamcunque æquationem compoſitam ad ſimplicem. Si enim in ſuperiori prima æquatione ponatur* $a - b = p$, *erit* $ax^2 - bx^2 = px^2$. *Item ponendo* $-b + c = q$, *erit* $-bx + cx = qx$, *demum ponendo* $abc = r$, *eadem æquatio transformabitur in hanc ſimpliciſſimam* $x^3 + px^2 + qx = r$. *Similiter in ſuperiori ſecunda æquatione fiat* $-a - b = -p$, *erit* $-ax - bx = -px$; *fiat* $ac - bc + mf = q$, *ea in hanc transformabitur* $x^2 - px = q$. *Quod quidem quanti ſit uſus, ex inferius dicendis ſatis conſtabit.*

COROLL. II. *Præſtat non raro omnes æquationis terminos ad unam partem transferre, ita ut omnes quantitates*

nibilo

nibilo fiant æquales. Sic æquatio $x^2 - px = q$ *fiet* $x^2 - px - q = 0$. *Item* $x^3 - ax^2 + bx = a^2b^2$, *erit* $x^3 - ax^2 + bx - a^2b^2 = 0$. *Quod artificium Algebræ, ut videbimus, valde commodum*, Thomæ Harrioto (a) *Anglo tribuitur.*

Coroll. III. *Si quis terminus in æquatione deſit, ut ſecundus, tertius, quartus &c. Notatur termini, aut terminorum (ſi plures deſint) defectus* aſteriſmo, *ut æquatio* $x^3 * + ax - a^3 = 0$, *quæ ſecundo termino caret. Pariter* $x^4 * - cx^2 * + a^2bc = 0$ *caret terminis ſecundo & quarto. Tunc autem minime variatur ſequentium terminorum ordo, aut conditio. Sed* ax *v.g. in prima æquatione manet terminus æquationis tertius; & in ſecunda* $+ a^2bc$ *tenet locum æquationis quintum, licet abſint duo termini, & ſic de aliis.*

PROPOSITIO III.

Radices veras in falſas, & falſas in veras commutare.

Mutentur in æquatione ſigna terminorum parium, ea nempe, quæ præcedunt terminos ſecundum, quartum, ſextum &c. radices veræ degenerabunt in falſas, & viciſſim falſæ in veras. Sit æquatio $x^3 - 3x^2 - 10x + 24 = 0$, quæ ex *anteced. Prop.* habet duas radices veras $+2$, $+4$, & unam falſam -3. Variatis autem terminorum parium ſignis, fit $x^3 + 3x^2 - 10x - 24 = 0$, quæ quidem habet eaſdem cum priore radices, ſed duas falſas -2, -4, & unam veram $+3$.

Demon-

(a) Artis Analyt. praxis Londini 1631.

Demonſtratio. Subſtituatur in eadem æquatione loco incognitæ x una ex radicibus, ſeu valoribus -2, tota æquatio evaneſcit, nempe $-8+12+20-24=0$. Subſtituatur deinde -4, iterum æquatio evaneſcit, nam $-64+48+40-24=0$. Idem fiat cum radice vera $+3$; erit enim $27+27-30-24=0$, ergo -2, -4, $+3$, ſunt radices illius æquationis *per Prop.* I. *hujus num.* 5. Quod &c.

PROPOSITIO IV.

Radices augere, vel minuere data quantitate.

I. Sit æquatio $x^3-3x^2-10x+24=0$, quæ *ex Prop.* I. *hujus* habet duas radices veras $+2$, $+4$, & unam falſam -3, augere volo ejus radices quantitate data $=3$.

Fiat $x+3=y$, erit $x=y-3$. Subſtituatur hic valor in æquatione loco ipſius x, erit

$$\begin{array}{r|rrrr} x^3 & =y^3 & -9y^2 & +27y & -27 \\ -3x^2 & & -3y^2 & +18y & -27 \\ -10x & & & -10y & +30 \\ +24 & & & & +24 \\ \hline \text{Summa} & y^3 & -12y^2 & +35y & +* =0 \end{array}$$

Hæc nova æquatio habet ſingulas radices ternario auctas, nempe $+5$, $+7$, $+0$ (nam $-3+3=0$) quæ quidem prius erant $+2$, $+4$, -3. Subſtituatur enim quælibet ex his in æquatione, ex. gr. $+5$, fiet $125-300+175=0$, ergo $+5$ eſt radix ejuſdem æquationis *per Propoſ.* I. *num.* 5.

 II. Mi-

II. Minuenda ſit eadem æquatio $x^3 - 3x^2 - 10x + 24 = 0$ numero binario. Fiat $x - 2 = y$, erit $x = y + 2$. Subſtituatur in eadem æquatione hic valor loco ipſius x, erit.

$$\begin{array}{r|rrrr} x^3 & = y^3 & + 6y^2 & + 12y & + 8 \\ -3x^2 & & - 3y^2 & - 12y & - 12 \\ -10x & & & - 10y & - 20 \\ +24 & & & & + 24 \\ \hline \text{Summa} & y^3 & + 3y^2 & - 10y & + * = 0 \end{array}$$

Hæc nova æquatio habet ſingulas radices binario minutas, nempe $0, -5, +2$, quæ prius erant $+2, -3, +4$. Subſtituatur enim quælibet ex his in æquatione, v. g. -5, erit $-125 + 75 + 50 = 0$, ergo *per Prop.* 1. *num.* 5. radix -5 eſt radix ejuſdem æquationis.

Coroll. I. *Augendo radices æquationis, ſingulæ radices veræ augentur. Contra vero falſæ minuuntur. Nam ſi ad radicem falſam* -4 *addatur* $+3$, *minuitur & fit* -1. *Imo aliquando falſæ in veras tranſeunt, ut ſi ad falſam* -4 *addas* $+5$, *fit radix vera* $+1$. *Quod evidens eſt.*

Coroll. II. *Minuendo radices æquationis, radices falſæ augentur. Nam ſi ex radice falſa* -3 *ſubtrahitur quantitas* $+2$, *fit* -5, *ut patet. At veræ minuuntur, imo & aliquando fiunt falſæ, ut ſi ex radice* $+2$ *auferas* $+5$, *fiet* -3, *radix ſcilicet falſa.*

Schol. *Augendo radices æquationis habetur methodus convertendi radices falſas in veras, nec propterea veræ fiant falſæ; quod docuit* Carteſius *(a), & patet ex* Cor. 1.

PRO-

(a) Geometriæ lib. 3. edit. 3. pag. m. 74.

PROPOSITIO V.

Ex data æquatione ſecundum terminum tollere.

Si ſecundus terminus datæ æquationis afficitur ſigno $+$, augeantur radices quantitate cognita ſecundi termini diviſa per exponentem æquationis. Si afficitur ſigno $-$, minuantur radices eadem quantitate; in utroque caſu habebitur nova æquatio ſecundo termino carens.

Sit æquatio $x^2 + 6x - 16 = 0$, ex qua ſecundus terminus tolli debeat.

Diviſa quantitate cognita ſecundi termini 6 per exponentem 2, quotus 3 eſt quantitas, qua augeri debent (ob ſignum $+$) radices æquationis.

Fiat ergo ut in *Prop. antec.* $x + 3 = y$, erit $x = y - 3$ quo valore ſubſtituto in æquatione, loco ipſius x, erit

$$\begin{array}{r|r} x^2 & = y^2 - 6y + 9 \\ +6x & +6y - 18 \\ -16 & -16 \\ \hline \text{Summa} & y^2 \ * \ - 25 = 0 \end{array}$$

Eſt ergo $y = 5$, proinde x $(= y - 3) = 5 - 3 = 2$.

II. Sit æquatio $x^3 - 12x^2 + 44x - 48 = 0$, in qua ſecundus terminus evaneſcere debeat.

Diviſa quantitate cognita ſecundi termini per exponentem 3, habetur $\frac{12}{3} = 4$, quantitas, qua minui debent (ob ſignum $-$) æquationis datæ radices. Fiat ergo $x - 4$

$=y$, erit $x=y+4$, factaque hujus valoris ſubſtitutione, oritur nova æquatio ſecundo termino carens

$$\begin{array}{r|rrrr}
x^3 & =y^3 & +12y^2 & +48y & +64 \\
-12x^2 & & -12y^2 & -96y & -192 \\
+44x & & & +44y & +176 \\
-48 & & & & -48 \\
\hline
\text{Summa} & y^3 & * & -4y & * = 0
\end{array}$$

III. Quid vero ſi ſecundus terminus ex æquatione tollendus utroque ſigno + & — afficitur?

Sit æquatio $x^2 \begin{smallmatrix} -ax \\ +bx \end{smallmatrix} -ab=0$, ex qua tolli debet ſecundus terminus. Diviſa quantitate cognita $-a+b$ per exponentem 2, habetur $\frac{-a+b}{2}$, ſeu $-\frac{1}{2}a+\frac{1}{2}b$.

Fiat $x-\frac{1}{2}a+\frac{1}{2}b=y$, erit $x=y+\frac{1}{2}a-\frac{1}{2}b$, factaque hujus valoris ſubſtitutione, habetur æquatio ſecundo termino carens

$$\begin{array}{r|l}
x^2 & =y^2+ay-by+\frac{1}{4}a^2-\frac{1}{2}ab+\frac{1}{4}b^2 \\
\begin{smallmatrix} -a \\ +b \end{smallmatrix}x & \quad -ay+by-\frac{1}{2}a^2+ab-\frac{1}{2}b^2 \\
-ab & \qquad\qquad -ab \\
\hline
\text{Summa} & y^2 \quad * \quad -\frac{1}{4}a^2-\frac{1}{2}ab-\frac{1}{4}b^2=0
\end{array}$$

Demonſtratur per Coroll. 1. & 2. *Propoſ. antec.* radices veræ ac falſæ ita augeri poſſunt ac minui, ut fiant æquales nullitati, ſeu zero. Sic in primo exemplo æquatio x^2
$+6x$

$+ 6x - 16 = 0$ habet duas radices, unam falſam $- 8$, alteram veram $+ 2$, proinde augendo numero ternario utranque, prima erit $- 8 + 3 = - 5$, altera vero $+ 2 + 3 = 5$, proinde $- 5 + 5 = 0$.

Similiter ſit æquatio $x^2 - 4x - 12 = 0$; ut tollatur ſecundus terminus, minui debent ejus radices numero binario. Una ipſius radix vera eſt $+ 6$, altera falſa $- 2$, proinde utranque minuendo, erit prima $+ 6 - 2 = 4$, altera $- 2 - 2 = - 4$, hinc $+ 4 - 4 = 0$. Atque hinc eſt, quod in æquatione, cujus omnes radices æquales ſunt, ſublato ſecundo termino, omnes alii evaneſcunt, ut videre eſt in æquatione $x^3 - 12x^2 + 48x - 64 = 0$, atque aliis. Qua ratione autem regula ſic augendi, vel minuendi radices, ſit inventa, patebit ex ſequenti *Propoſ. ejuſque Schol.* 1.

SCHOL. I. *Cum ſæpe diviſa quantitate cognita ſecundi termini per exponentem primi, oriatur fractio, ut in tertio exemplo ſuperiori contigit, quo praxis facilior in æquationibus præſertim altioribus evadat, præſtat loco fractionis indeterminatam aliquam aſſumere, & binomium* $x + n$, *vel* $x - n$ *elevare ad omnes illos gradus, quos æquatio data exigit; peractaque operatione, valores indeterminatæ* n^1, n^2, n^3 *&c. ſubſtituere. Sic enim moleſtia fractionum minuitur, & facilius poterit ad examen operatio tota revocari. Data ſit æquatio, ex qua ſecundus terminus auferri debeat,* $x^3 + 5x^2 - 2x - 24 = 0$. *Diviſa quantitate cognita ſecundi termini per exponentem* 3, *habetur fractio* $\frac{5}{3}$, *fiat* $n = \frac{5}{3}$, *&* $x + n = y$, *erit* $x = y - n$, *quo quidem binomio ad ſecundam, & tertiam poteſtatem elevato, hoc eſt*

$$x = y - n$$
$$x^2 = y^2 - 2ny + n^2$$
$$x^3 = y^3 - 3ny^2 + 3n^2y - n^3$$

Poſitiſque loco n, n^2, n^3 *eorum valoribus* $n = \frac{5}{3}$, $n^2 = \frac{25}{9}$ $n^3 = \frac{125}{27}$, *facili negotio invenitur.*

$$\begin{array}{r|l} x^3 & = y^3 - 5y^2 + \frac{75}{9}y - \frac{125}{27} \\ + 5x^2 & \quad\quad + 5y^2 - \frac{50}{3}y + \frac{125}{9} \\ - 2x & \quad\quad\quad\quad - 2y + \frac{10}{3} \\ - 24 & \quad\quad\quad\quad\quad\quad - 24 \\ \hline \text{Summa} & y^3 \; * \; - \frac{93}{9}y - \frac{308}{27} = 0 \end{array}$$

Schol. II. *In omni æquatione terminus ſecundus, qui deficit, ſupponi poteſt affectus utroque ſigno* + *&* —; *ſic æquatio* $x^3 \; * \; - ax + b^3 = 0$ *ſupponitur* $x^3 \pm - ax + b^3 = 0$, *proinde facta comparatione cum termino, qui præcedit, nempe cum* $+ x^3$, *habetur & permutatio & ſucceſſio ſignorum, hoc eſt* + — *&* + +. *Quod pariter accidit, facta comparatione cum termino, qui conſequitur, nempe cum* — ax. *Nam ſimiliter habetur* + — *&* — —; *ideoque in utroque caſu indicatur una radix vera, & altera falſa* per Prop. 1. hujus num. 3.

PROPOSITIO VI.

Ex æquatione terminum tertium tollere.

I. Sit æquatio $x^3 + px^2 + qx + r = 0$, ex qua tertius terminus ſit auferendus.

Inve-

Inveniri debet quantitas, qua radices datæ æquationis ſic augeantur, aut minuantur, ut tertius terminus evaneſcat, quod quidem per analyſim obtinetur hoc artificio. Sit quantitas quæſita $= z$; fiat $x + z = y$, erit $x = y - z$; ſubſtituto hoc valore ipſius x in data æquatione, erit

$$\left.\begin{array}{r} x^3 \\ +px^2 \\ +qx \\ -r \end{array}\right| \begin{array}{rrrr} = y^3 - 3zyy & + 3z^2 y & - z^3 \\ + pyy & - 2pzy & + pz^2 \\ & + qy & - qz \\ & & - r \end{array}$$

Cum igitur tertius terminus debeat evaneſcere, fiat æqualis zero, ſcilicet

$$\begin{array}{ll} & 3z^2y - 2pzy + q = 0 \\ \textit{Divid. per } y & 3z^2 - 2pz = -q \\ \textit{Tum per } 3 & z^2 - \tfrac{2}{3}pz = -\tfrac{1}{3}q \end{array}$$

Occurrit æquatio ſecundi gradus reſolvenda, ut docebitur infra *Prop.* 1. *Cap.* 8. quæ ab hac non dependet. Addatur ſcilicet utrinque quadratum ex dimidio coefficientis ſecundi termini, nempe $\frac{1}{9}pp$, erit

$$z^2 - \tfrac{2}{3}pz + \tfrac{1}{9}pp = \tfrac{1}{9}pp - \tfrac{1}{3}q$$

$$\textit{Extr. Rad.} \quad z - \tfrac{1}{3}p = \sqrt{\tfrac{1}{9}pp - \tfrac{1}{3}q}$$

$$z = \tfrac{1}{3}p + \sqrt{\tfrac{1}{9}pp - \tfrac{1}{3}q}$$

Sed $x = y - z$, ergo $x = y - \frac{1}{3}p - \sqrt{\frac{1}{9}pp - \frac{1}{3}q}$. Quo valore in data æquatione ſubſtituto, fiet æquatio termino tertio carens, ut in duobus ſequentibus exemplis.

1. Sit

1. Sit enim æquatio $x^3 - 4x^2 + 4x - 6 = 0$, erit $p = -4$ & $q = 4$, hinc $y - \frac{1}{3}p - \sqrt{\frac{1}{9}pp - \frac{1}{3}q} = y + \frac{4}{3} - \frac{2}{3} = y + \frac{2}{3}$. Eſt ergo $x = y + \frac{2}{3}$, ideoque erit

$$\begin{array}{r|rrrr}
x & = y^3 & + 2y^2 & + \frac{4}{3}y & + \frac{8}{27} \\
-4x^2 & & -4y^2 & - \frac{16}{3}y & - \frac{16}{9} \\
+4x & & & + 4y & + \frac{8}{3} \\
-6 & & & & -6 \\
\hline
\text{Summa} & y^3 & - 2y^2 & * & - \frac{130}{27} = 0
\end{array}$$

2. Similiter ſit æquatio data $x^3 - 3x^2 + 3x - 4 = 0$, erit $p = -3$ & $q = 3$, proinde $y - \frac{1}{3}p - \sqrt{\frac{1}{9}pp - \frac{1}{3}q} = y + 1$. Fiat cum hoc valore nova æquatio, nempe

$$\begin{array}{r|rrrr}
x^3 & = y^3 & + 3y^2 & + 3y & + 1 \\
-3x^2 & & -3y^2 & - 6y & - 3 \\
+3x & & & + 3y & + 3 \\
-4 & & & & -4 \\
\hline
\text{Summa} & y^3 & * & * & - 3 = 0
\end{array}$$

Habetur ergo æquatio termino tertio carens, in qua $y = \sqrt[3]{3}$.

II. Sit æquatio quarti gradus, ex qua tertius terminus exterminari debeat, $x^4 \; * + px^2 + qx + r = 0$.

Fiat $x + z = y$, erit $x = y - z$, quo valore ipſius x in æquatione data ſubſtituto, habetur

x^4

$$
\begin{array}{r|l}
x^4 & = y^4 - 4zy^3 + 6z^2y^2 - 4z^3y + z^4 \\
+px^2 & \qquad\qquad\quad + py^2 \quad - 2pzy + pz^2 \\
+qx & \qquad\qquad\qquad\qquad\quad + qy - qz \\
+r & \qquad\qquad\qquad\qquad\qquad\quad + r
\end{array}
$$

Fiat terminus $6z^2y^2 + py^2 = 0$; dividendo per y^2, erit $6z^2 + p = 0$, hoc eſt $z^2 = -\frac{p}{6}$, & $z = \sqrt{-\frac{p}{6}}$; habetur ergo valor ipſius z.

Data ſit jam æquatio ſpecialis $x^4 * -6x^2 - 8x - 3 = 0$, ex qua tertius terminus auferendus ſit. Facta comparatione terminorum hujus cum terminis æquationis ſuperioris, erit $p = -6$, $q = -8$ &c. adeoque z $(= \sqrt{-\frac{p}{6}})$ $= \sqrt{\frac{6}{6}} = 1$. Eſt ergo $x = y - 1$, qui valor ſi ponatur in æquatione data, oritur

$$
\begin{array}{r|l}
x^4 & = y^4 - 4y^3 + 6y^2 - 4y + 1 \\
-6x^2 & \qquad\qquad\quad - 6y^2 + 12y - 6 \\
-8x & \qquad\qquad\qquad\quad - 8y + 8 \\
-3 & \qquad\qquad\qquad\qquad\quad - 3 \\
\hline
\text{Summa} & \quad y^4 - 4y^3 \quad * \quad * \quad * = 0
\end{array}
$$

Unde æquatio facillime reſolvitur. Nam $y^4 = 4y^3$, proinde $y = 4$. Sed poſita fuit $x = y - z$, ergo $x = 4 - 1 = 3$; eſt ergo 3 radix ipſius æquationis.

SCHOL. I. *Hoc eodem artificio inventa eſt regula tollendi ex æquationibus terminum ſecundum, quam in* Prop. ant. *docuimus. Nam ſi ex æquatione generali ſuperiori ſecundus terminus ſupponatur æqualis zero, erit* $-3zyy +$

pyy = o, *proinde* 3z = p, & z = $\frac{p}{3}$; *unde patet, quantitatem addendam, vel ſubtrahendam radici datæ æquationis eſſe quantitatem cognitam ſecundi termini diviſam per exponentem primi termini.*

Schol. II. *Eadem plane ratione tolli poſſunt ex æquationibus termini* quartus, quintus, ſextus &c. *Sed quia pro quarto tollendo æquatio tertii gradus, pro quinto æquatio quarti gradus, pro ſexto æquatio quinti gradus occurrit, ideo non eſt hujus loci ulterius progredi.*

PROPOSITIO VII.

Æquationem, in qua termini aliqui deſunt, complere.

I. Augeatur radix æquationis datæ quantitate aliqua *per Prop.* 4. exurget æquatio completa.

Sit æquatio complenda $x^3 - 19x - 30 = 0$, in qua deeſt ſecundus terminus. Fiat *per Prop. cit.* $x + 1 = y$, erit $x = y - 1$. Subſtituatur hic valor loco x in data æquatione, erit

$$\begin{array}{r|l} x^3 & = y^3 - 3y^2 + 3y - 1 \\ -19x & \qquad\quad - 19y + 19 \\ -30 & \qquad\qquad\quad -30 \\ \hline \text{Summa} & y^3 - 3y^2 - 16y - 12 = 0 \end{array}$$

Habetur ergo æquatio completa cum ſecundo termino, in qua $y = x + 1$.

II. Sit

II. Sit æquatio incompleta $x^3 - 3x^2 + 2 = 0$, in qua tertius terminus deficit. Fiat, ut ſupra, $x + 1 = y$, erit $x = y - 1$, & facta ſubſtitutione hujus valoris loco ipſius x, erit

x^3	$= y^3 - 3y^2 + 3y - 1$
$-3x^2$	$-3y^2 + 6y - 3$
$+2$	$+2$
Summa	$y^3 - 6y^2 + 9y - 2 = 0$

En integra æquatio, in qua pariter $y = x + 1$.

Ratio eſt, quia ſecundus terminus v. g. deficit in æquatione, cum radices veræ æquantur falſis *per Prop.* 1. *num.* 4. ergo ſi una ex veris augeatur, integra æquatio reſtituitur.

PROPOSITIO VIII.

Æquationis radices per datam quantitatem multiplicare.

I. Eſt æquatio $x^3 - ax^2 - b^2 x + abb = 0$, cujus radices $+a$, $+b$, $-b$ multiplicandæ ſunt per datam quantitatem $= c$.

Fiat $cx = y$, erit $x = y : c$ (duo puncta (:) ſunt diviſionis ſignum. *ex Schol.* 3. *Prop.* 8. *Cap.* 1) Subſtituatur ubique hic valor in data æquatione loco ipſius x, erit nova æquatio

$$\frac{y^3}{c^3} - \frac{ay^2}{c^2} - \frac{b^2 y}{c} + abb = 0$$

Et multiplicatis ſingulis terminis per c^3, factaque reductione, habetur.

$$y^3 - acy^2 - bbc^2y + abbc^3 = 0$$

cujus radices ſunt $+ac$, $+bc$, $-bc$.

II. Similiter eſt æquatio $x^3 - 6x^2 + 11x - 6 = 0$, cujus radices $+1, +2, +3$, multiplicari debent per quantitatem $= 2$.

Fiat $2x = y$, erit $x = y : 2$, & hoc valore ſubſtituto in data æquatione loco ipſius x, oritur

$$\frac{y^3}{8} - \frac{6y^2}{4} + \frac{11y}{2} - 6 = 0$$

Multiplicentur omnes termini per cubum 8, factaque reductione, erit nova æquatio

$$y^3 - 12y^2 + 44y - 48 = 0$$

cujus radices ſunt $+2$, $+4$, $+6$.

Corol. I. *Ex utroque exemplo ſatis apparet, ad multiplicandam æquationem per datam quantitatem, ſatis eſſe eam multiplicare per progreſſionem Geometricam, cujus terminus primus ſit* 1, *terminus ſecundus ſit denominator rationis &c. Nam multiplicandæ ſint radices ejuſdem prioris æquationis per* c, *aſſumpta alia incognita* y *loco ipſius* x *erit, ut prius*

$$\begin{array}{cccc} y^3 - & ay^2 - & b^2y + & abb = 0 \\ 1. & c. & c^2. & c^3. \\ \hline y^3 - & acy^2 - & b^2c^2y + & abbc^3 = 0 \end{array}$$

Simi-

Similiter multiplicandæ ſunt per 2 radices alterius æquationis: aſſumpta y loco ipſius x, & multiplicata æquatione per progreſſionem Geometricam, erit ut prius.

$$\begin{array}{l} y^3 - 6y^2 + 11y - 6 = 0 \\ 1. \quad * \quad 2. \quad 4. \quad 8. \\ \hline y^3 - 12y^2 + 44y - 48 = 0 \end{array}$$

PROPOSITIO IX.

Radices æquationis dividere per datam quantitatem.

I. Sit æquatio $x^3 - ax^2 - b^2x + abb = 0$, cujus radices $+a$, $+b$, $-b$ dividere oporteat per quantitatem $= c$.

Fiat $\frac{x}{c} = y$, erit $x = cy$, & ſubſtituto hoc valore in data æquatione, erit

$$c^3y^3 - ac^2y^2 - b^2cy + abb = 0$$

Dividantur ſinguli termini per c^3, habetur

$$y^3 - \frac{ay^2}{c} - \frac{b^2y}{c^2} + \frac{abb}{c^3} = 0$$

cujus radices ſunt $+\frac{a}{c}$, $+\frac{b}{c}$, $-\frac{b}{c}$.

COROL. *Hinc apparet, ad dividendam æquationem per datam quantitatem, ſatis eſſe eam dividere per progreſ-*

ſionem Geometricam, cujus primus terminus ſit 1, *ſecundus terminus ſit denominator rationis. Nam aſſumpta nova incognita* y, *& progreſſione Geometrica* $1. c. c^2. c^3$ *&c. erit æquatio, ut prius, diviſa per* c, *nempe*

$$\frac{\begin{matrix} y^3 - ay^2 - b^2y + abb = 0 \\ 1. \quad c. \quad c^2. \quad c^3. \end{matrix}}{y^3 - \frac{ay^2}{c} - \frac{b^2y}{c^2} + \frac{abb}{c^3} = 0.}$$

II. Eadem ratione dividenda ſit per 3 æquatio numerica ſuperior $x^3 - 6x^2 + 11x - 6 = 0$, cujus radices ſunt $+1, +2, +3$. Dividatur æquatio (aſſumpta incognita y) per progreſſionem Geometricam 1, 3, 9 &c. nempe

$$\frac{\begin{matrix} y^3 - 6y^2 + 11y - 6 = 0 \\ 1. \quad 3. \quad 9. \quad 27. \end{matrix}}{y^3 - 2y^2 + \frac{11y}{9} - \frac{2}{9} = 0}$$

Habetur nova æquatio, cujus radices ſunt $\frac{1}{3}, \frac{2}{3}$, 1. Nam harum qualibet ſubſtituta in æquatione loco ipſius y, tota æquatio evaneſcit, proinde ſunt radices quæſitæ *per Prop.* 1. *hujus num.* 5.

Schol. *Ut æquationum radices augeantur, minuantur, multiplicentur, aut dividantur ea ratione, quam docuimus, plane non opus eſt, ut illæ ſint cognitæ, imo a Carteſio, & aliis tanquam prorſus incognitæ ſupponuntur. Nos tamen conſulto, illuſtrandæ rei gratia, ut jam cognitas uſurpavimus.*

PRO-

PROPOSITIO X.

Æquationem a fractionibus liberare.

I. Multiplicetur radix æquationis per factum omnium denominatorum fractionum occurrentium, aut per numerum, qui omnes denominatores metitur.

Sit æquatio data $x^3 - \frac{ax^2}{b} + \frac{aax}{c} + \frac{a^3}{d} = 0$

Multiplicetur per factum omnium denominatorum bcd radix æquationis; hoc eſt fiat $bcdx = y$, erit $x = y : bcd$, factaque hujus valoris ſubſtitutione in data æquatione, oritur

$$\frac{y^3}{b^3c^3d^3} - \frac{ay^2}{b^3c^2d^2} + \frac{aay}{bc^2d} + \frac{a^3}{d} = 0$$

Multiplicentur ſinguli termini per $b^3c^3d^3$, habetur æquatio ſine fractionibus

$$y^3 - acdy^2 + a^2b^2cd^2y + a^3b^3c^3d^2 = 0.$$

II. Similiter ſit æquatio $x^3 - \frac{1}{2}x^2 + \frac{1}{5}x - 24 = 0$. Multiplicetur radix æquationis per factum ex denominatoribus, hoc eſt per 10, nempe per progreſſionem Geometricam 1. 10, &c. *per Coroll. Prop.* 8. *hujus.*

$$x^3 - \frac{1}{2}x^2 + \frac{1}{5}x - 24 = 0$$
$$1. \quad 10. \quad 100. \quad 1000.$$
$$y^3 - 5y^2 + 20y - 24000 = 0$$

In qua quidem æquatione $y = 10x$, proinde erit $x = \frac{y}{10}$.

III. De-

III. Demum ſit æquatio $x^3 * \frac{2}{3}x - \frac{18}{27} = 0$. Quia quantitas 3 metitur utrumque denominatorem, multiplicetur radix æquationis per 3 modo explicato, nempe

$$\begin{array}{l} x^3 \; * \; \frac{2}{3}x - \frac{18}{27} = 0 \\ 1. \; 3. \; 9. \quad 27. \\ \hline y^3 \; * \; 6y - 81 = 0 \end{array}$$

In hac æquatione habetur $y = 3x$, proinde dividi debent per 3 ſingulæ radices hujus, ut habeantur radices æquationis datæ, quod ſemper advertendum.

PROPOSITIO XI.

Æquationem a radicalibus liberare.

ID non eadem via ſemper aſſequimur; ideoque plura exempla rem illuſtrabunt.

I. Sit æquatio $x^3 * - 5x + \sqrt{2} = 0$. Fiat $x : \sqrt{2} = y$, erit $x = y\sqrt{2}$, & hoc valore in data æquatione ſubſtituto, erit

$$\begin{array}{rcl} x^3 & = & 2y^3\sqrt{2} \\ -5x & = & -5y\sqrt{2} \\ +\sqrt{2} & = & +\sqrt{2} \\ \hline \text{Summa} & & 2y^3\sqrt{2} - 5y\sqrt{2} + \sqrt{2} \end{array}$$

Et diviſis omnibus terminis per $\sqrt{2}$, fit æquatio ſine radicalibus

$$2y^3 - 5y + 1 = 0.$$

II. Quandoque fit progreſſio Geometrica ex radicalibus, per quam tota æquatio multiplicata ab irrationalibus liberatur. Ecce exemplum

x^4

$$x^4+2ax^3\sqrt{2}+5abx^2-a^3x\sqrt{8}-2a^2b^2=0$$
$$1 \quad \sqrt{2}. \quad 2. \quad \sqrt{8}. \quad 4.$$
$$y^4+4ay^3+10aby^2-8a^3y-8a^2b^2=0$$

Radices autem hujus æquationis dividendæ erunt per $\sqrt{2}$, ut habeantur radices æquationis datæ. Nam $y=x\sqrt{2}$.

III. Sæpe etiam tota æquatio dividitur per radicalium progreſſionem Geometricam.

$$x^3-ax^2\sqrt[3]{2}+abx\sqrt[3]{32}-a^2b=0$$
$$1. \quad \sqrt[3]{2}. \quad \sqrt[3]{4}. \quad 2.$$
$$y^3-ay^2+2aby-\tfrac{1}{2}a^2b=0$$

In hac æquatione eſt $y=x:\sqrt[3]{2}$.

IV. Radicales ex æquationibus aliquando exterminantur per multiplicationem. Sit æquatio

$$x=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}\times\sqrt[3]{\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}=0$$

Multiplicentur radices univerſales *per Prop.* 12. *Cap.* 4.

$$\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}$$
$$\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}$$
$$\tfrac{1}{4}qq+\tfrac{1}{2}q\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}$$
$$-\tfrac{1}{2}q\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}-(\tfrac{1}{4}qq-\tfrac{1}{27}p^3)$$
$$\tfrac{1}{4}qq \quad * \quad * \quad -\tfrac{1}{4}qq+\tfrac{1}{27}p^3=\sqrt[3]{\tfrac{1}{27}p^3}$$

$=\tfrac{1}{3}p$, hinc $x=\tfrac{1}{3}p$. Ex $-1\times\tfrac{1}{4}qq-\tfrac{1}{27}p^3$ oritur $-\tfrac{1}{4}qq+\tfrac{1}{27}p^3$.

V. Æquatio a radicalibus liberatur ſupponendo terminos radicales æquales literis ad libitum aſſumptis, quæ ſingulatim in uno æquationis membro collocantur, & tota æquatio elevatur ad potentiam ab exponente radicis indicatam. Sit æquatio

$$x^3 * - x\sqrt{a} + \sqrt{b} = 0$$

Fiat $\sqrt{a} = p$, & $\sqrt{b} = q$, æquatio in hanc transformatur

$$x^3 * - px + q = 0$$

Ponatur px ſeorſim in uno æquationis membro, nempe $px = x^3 + q$, & (ob $\sqrt{a} = p$) tota æquatio elevetur ad ſecundam poteſtatem *per Reg.* 4. *Cap.* 5.

$$p^2 x^2 = x^6 + 2qx^3 + q^2$$

Deinde ponatur $2qx^3$ in uno æquationis membro, hoc eſt $2qx^3 = p^2 x^2 - x^6 - q^2$, & iterum utrunque membrum ad ſecundam poteſtatem elevetur ob $\sqrt{b} = q$; *per Reg. cit.* erit

$$4q^2 x^6 = p^4 x^4 - 2p^2 x^8 + x^{12} - 2p^2 q^2 x^2 + 2q^2 x^6 + q^4$$

hoc eſt $x^{12} - 2p^2 x^8 - 2q^2 x^6 + p^4 x^4 - 2p^2 q^2 x^2 + q^4 = 0$

Quod ſi fiat $x^2 = y$, deprimi poterit ad æquationem ſexti gradus, ſcilicet

$$y^6 - 2p^2 y^4 - 2q^2 y^3 + p^4 y^2 - 2p^2 q^2 y + q^4 = 0$$

Demum in hac ultima æquatione loco p^2, p^4, q^2, q^4, ſubſtituantur eorum valores, nempe a, a^2, b, b^2 obtinebitur æquatio a radicalibus expedita.

$$y^6 - 2ay^4 - 2by^3 + a^2 b^2 - 2aby + b^2 = 0$$

Schol.

SCHOL. *Si radicales ſint cubicæ, æquationis membra ad tertiam poteſtatem elevantur.*

PROPOSITIO. XII

Æquationem datam in aliam commutare, in qua quantitas cognita cujuſcunque termini, vel etiam terminus ultimus fiat datæ quantitati æqualis.

I. INveniri debet quantitas, per quam ita multiplicentur æquationis datæ radices, ut quantitas cognita in aliam datam commutetur.

Sit æquatio $x^3 - px^2 + qx - r = 0$, quantitas data $= a$, quæſita ſit $= z$.

Fiat $x = \frac{y}{z}$, & hic valor ponatur in data æquatione loco ipſius x, aſſumpta nova incognita y, erit

$$y^3 - pzy^2 + qz^2y - rz^3 = 0$$

1. Fieri debeat quantitas cognita ſecundi termini $p = a$: ſi ſupponatur $pz = a$, erit $z = \frac{a}{p}$.

2. Quantitas cognita tertii termini q fieri debeat æqualis datæ quantitati a, erit $qz^2 = a$, proinde $z^2 = \frac{a}{q}$, atque hinc $z = \sqrt{\frac{a}{q}}$.

3. Pari modo fieri debeat ultimus terminus $rz^3 = a$, erit $z^3 = \frac{a}{r}$, ideoque $z = \sqrt[3]{\frac{a}{r}}$.

Jam ſi multiplicetur ſecundus terminus cujuſcunque æquationis per valorem inventum ipſius z, tertius terminus per quadratum, quartus per cubum ejuſdem valoris, & ſic deinceps, habebitur æquatio quæſita. Tyronum gratia res exemplis illuſtratur.

Sit æquatio $x^3 - 3x^2 + 18x - 54 = 0$, quæritur loco ipſius alia, in qua coefficiens ſecundi termini 3 ſit $= 2$. Erit ergo $p = 3$, $a = 2$; & cum ſit ex dictis $z = \frac{a}{p}$, erit $z = \frac{2}{3}$, proinde ſi per hunc valorem multiplicentur radices hujus æquationis, aſſumpta nova incognita y, ut moris eſt, habebitur

$$\begin{array}{c} x^3 - 3x^2 + 18x - 54 = 0 \\ \quad\ \frac{2}{3} \qquad \frac{4}{9} \qquad \frac{8}{27} \\ \hline y^3 - 2y^2 + 8y - 16 = 0 \end{array}$$

II. Sit æquatio $x^3 - 12x^2 + 9x - 27 = 0$, cujus loco quæritur alia, in qua quantitas cognita tertii termini ſit 4. Erit $q = 9$, $a = 4$.

Quia vero $z = \sqrt{\frac{a}{q}}$ ex dictis ſuperius, erit $z = \sqrt{\frac{4}{9}} = \frac{2}{3}$, per hanc igitur quantitatem mutiplicandæ ſunt æquationis radices, nova incognita y aſſumpta, hoc eſt

$$\begin{array}{c} x^3 - 12x^2 + 9x - 27 = 0 \\ \quad\ \frac{2}{3} \qquad \frac{4}{9} \qquad \frac{8}{27} \\ \hline y^3 - 8y^2 + 4y - 8 = 0 \end{array}$$

III.

III. Sit pro tertio caſu eadem æquatio $x^3 - 12x^2 + 9x - 27 = 0$, quæ in aliam commutanda, cujus terminus ultimus ſit $= 1$.

Erit $r = 27$; $a = 1$: quia vero ex annotatis ſuperius $z = \sqrt[3]{\frac{a}{r}}$, erit $z = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, per quam quidem quantitatem multiplicari debent radices datæ æquationis, ut factum vides, & nova incognita y ſurrogari.

$$\begin{array}{l} x^3 - 12x^2 + 9x - 27 = 0 \\ \quad\;\; \frac{1}{3} \quad\;\; \frac{1}{9} \quad \frac{1}{27} \\ \hline y^3 - 4y^2 + y - 1 = 0 \end{array}$$

COROL. *Nemo non videt, hujus propoſitionis myſterium in eo eſſe, ut per Analyſim inveniatur incognitæ illius z aſſumptæ quantitas, per quam multiplicatis datæ cujuſcunque æquationis radicibus, alia nova oritur æquatio, in qua terminorum coefficientes, ſive etiam ultimus terminus, ſunt datæ quantitati æquales.*

PROPOSITIO XIII.

Invenire maximum duarum æquationum diviſorem communem.

I. MAjor æquatio, ea nempe, quæ altioris gradus incognitam continet (ſi ſint æqualis gradus, utravis per aliam) dividatur per minorem, omnino ut fit, in

aliis quantitatibus compoſitis *per Prop.* 8. *Cap.* 1. & neglecto quoto, continuetur diviſio, donec incognita in reſiduo fiat minoris dimenſionis, quam in diviſore. Tunc enim diviſor ipſe dividatur per illud reſiduum, neglecto quoto; & ſic ſemper, donec nihil ex diviſione remaneat. Nam talis diviſor erit maximus diviſor communis, qui quæritur, per quem æquationes illæ datæ ſunt diviſibiles. Exemplis res illuſtratur.

Sint duæ æquationes $3x^3 - 12x^2 + 15x - 6 = 0$, & $-12x^2 + 30x - 18 = 0$, quarum maximus diviſor communis quæritur.

Cum utraque ſit multiplicata per 3, dividi poterit per 3, ut termini minores fiant; erunt $x^3 - 4x^2 + 5x - 2 = 0$, & $-4x^2 + 10x - 6 = 0$, imo hæc adhuc dividi poteſt per 2, unde fit $-2x^2 + 5x - 3 = 0$.

Quia vero dividendo primum terminum x^3 per $-2x^2$, quotus eſt $\frac{x^3}{-2x^2} = \frac{x}{-2}$, hinc apparet, æquationem illam actu dividi non poſſe, quin prius multiplicetur per denominatorem -2. Quo facto, habetur $-2x^3 + 8x^2 - 10x + 4 = 0$. Hæc autem diviſa per $-2x^2 + 5x - 3$ (neglecto quoto) dat reſiduum $+3x^2 - 7x + 4$.

Continuetur diviſio, & quia dividendo $3x^2$ per $-2x^2$, quotus eſt $= \frac{3}{-2}$, ſignum eſt, æquationem dividendam prius multiplicari debere per denominatorem -2, unde fit $-6x^2 + 14x - 8 = 0$; factaque diviſione, habetur reſiduum $-x + 1$, ſeu $x - 1 = 0$; per quod dividendo quantitatem ipſam, quæ hactenus fuit diviſor, ſcilicet $-2x^2$

$+5x$

$+5x-3$, nihil remanet. Eſt ergo $x-1$ maximus diviſor communis duarum datarum æquationum, qui tamen multiplicari debet per eandem quantitatem, per quam primo æquatio utraque communiter fuit diviſa, eritque $3x-3=0$.

II. Sint duæ æquationes $x^3-16x^2+61x-66=0$, & $x^3+3x^2-34x+48=0$, quarum maxima communis menſura quæritur. Dividatur prima per ſecundam, & neglecto quoto 1, per reſiduum $-19x^2+95x-114$ (quod, ut ſimplicius evadat, dividitur per -19, fitque x^2-5x+6) dividatur ſecunda æquatio data: neglecto quoto x, habetur reſiduum $8x^2-40x+48$, quod diviſum per 8, dabit x^2-5x+6. Hoc autem cum ſit idem ac ſuperius reſiduum, ſi fiat diviſio, nihil remanet. Sunt ergo datæ æquationes diviſibiles per x^2-5x+6 maximum earum diviſorem communem.

III. Sint duæ æquationes $x^4-4ax^3+11a^2x^2-20a^3x+12a^4=0$, & $x^4-3ax^3+12a^2x^2-16a^3x+24a^4=0$. Dividatur prima per ſecundam, habetur 1 pro quoto: quo neglecto, remanet $-ax^3-a^2x^2-4a^3x-12a^4$, quo diviſo per $-a$, habetur primum reſiduum $x^3+ax^2+4a^2x+12a^3$.

Per hoc primum reſiduum dividatur ſecunda æquatio data, habetur quotus $x-4a$: quo neglecto, remanet $12a^2x^2-12a^3x+72a^4$, atque hoc diviſo per $12a^2$, habetur ſecundum reſiduum $x^2-ax+6a^2$. Per hoc ſecundum reſiduum dividatur primum, nihil remanet. Maximus ergo diviſor communis eſt $x^2-ax+6a^2$.

COROL. I. *Ex primo exemplo conſtat, communem diviſorem inventum tunc ſolum eſſe multiplicandum per quanti-*

tita-

titatem, per quam diviſa fuit æquatio, cum non una tantum, ſed utraque æquatio per communem quantitatem diviſa fuerit. Sic x — 1 *multiplicatur quidem per* 3, *per quem diviſa fuerat utraque æquatio, non autem per* 2, *per quem una tantum æquatio fuit diviſa.*

CorOL. II. *Reperto communi duarum æquationum diviſore, habentur pariter earum radices, & problematis reſolutio. Nam diviſoris communis radices ſunt eædem ac radices earundem æquationum. Sic in ſecundo exemplo diviſor communis, ſeu æquatio* $x^2 - 5x + 6 = 0$ *habet radices* 3 & 2, per Prop. 1. hujus num. 5., *quas datarum æquationum eaſdem eſſe facile intelligitur.*

Schol. I. *Si æquationes fuerint plures, quam duæ, reperto communi diviſore inter primam & ſecundam, inveniri deinde eodem modo debet diviſor communis inter tertiam æquationem & diviſorem communem jam inventum. Sed hoc perraro accidit.*

Schol. II. *Hanc regulam eandem eſſe ac regulam communis Arithmeticæ, qua invenitur maxima communis menſura, ſeu diviſor inter duos numeros datos, iiſdemque principiis inniti, nemo non videt.*

PROPOSITIO XIV.

Duarum æquationum diviſorem communem alia ratione inveſtigare.

I. SInt duæ æquationes Propoſ. præc. quarum diviſor communis quæritur, nempe *A* & *B*

$$\begin{array}{ll} A & x^4 - 4ax^3 + 11a^2x^2 - 20a^3x + 12a^4 = 0 \\ B & x^4 - 3ax^3 + 12a^2x^2 - 16a^3x + 24a^4 = 0 \end{array}$$

Sumatur ex A primi termini valor *per Coroll. 1. Prop. 1. Cap. 5.* erit $x^4 = 4ax^3 - 11a^2x^2 + 20a^3x - 12a^4$, qui ſurrogetur in æquatione B loco x^4, habebitur, facta terminorum inutilium reductione, æquatio

$$\begin{array}{rl} \textit{Div. per } a) & ax^3 + a^2x^2 + 4a^3x + 12a^4 = 0 \\ C & x^3 + ax^2 + 4a^2x + 12a^3 = 0 \end{array}$$

Valor ipſius x^3 ex hac æquatione C ſumptus ponatur in æquatione A, non ſolum loco ſecundi termini $-4ax^3$, ſed etiam loco primi x^4 (multiplicando $-ax^2 - 4a^2x - 12a^3 \times x$, ut fiat $= x^4$) erit primo $x^4 = -ax^3 - 4a^2x^2 - 12a^3x$; ſurrogato deinde valore ejusdem x^3 tam pro $-ax^3$ hujus æquationis, quam pro $-4ax^3$ æquationis A (hoc eſt pro $-5ax^3$) ſcilicet $5a^2x^2 + 20a^3x + 60a^4$, æquatio prima transformatur in ſequentem

$$\begin{array}{r|l} x^4 & = -4a^2x^2 - 12a^3x \\ -5ax^3 & \quad + 5a^2x^2 + 20a^3x + 60a^4 \\ 11a^2x^2 + 20a^3x & \quad + 11a^2x^2 - 20a^3x \\ + 12a^4 & \qquad\qquad\qquad + 12a^4 \\ \hline \text{Summa} & 12a^2x^2 - 12a^3x + 72a^4 = 0 \\ \textit{Divid. per } 12a^2) & x^2 - ax + 6a^2 = 0 \end{array}$$

Sumatur ex hac poſtrema æquatione valor primi termini, erit *per Coroll. cit.* $x^2 = ax - 6a^2$, qui ſurrogetur ubique in tertia æquatione C, etiam pro primo termino (ducendo $ax - 6a^2 \times x$, ut fiat ad x^3 æqualis) factaque terminorum ſubſtitutione, invenitur

$$6a^2x - 6a^2x + 12a^3 - 12a^3 = 0$$

Cumque termini omnes per ſigna contraria ſeſe deſtruant, ſignum eſt, æquationem $x^2 - ax + 6a^2 = 0$, ex qua terminorum contrarietas iſta derivatur, eſſe communem diviſorem maximum datarum æquationum, qualem prorſus in ſecundo exemplo *Prop. præc.* invenimus.

II. Quæritur diviſor communis duarum æquationum, quæ ſequuntur *M* & *N*.

$$M \quad x^4 + 4x^3 + 10x^2 + 14\tfrac{1}{2}x + 5 = 0$$
$$N \quad x^3 + 4x^2 + 8x + 8 = 0$$

Sumatur ex ſecunda valor primi termini *per Coroll. 1. Prop. 1. Cap. 5.* nempe

$$x^3 = -4x^2 - 8x - 8$$

Quo poſito in prima æquatione (multiplicando illum per x, ut fiat ad x^4 æqualis) habetur $2x^2 + 6\frac{1}{2}x + 5 = 0$, & multiplicando per 2 ad fractionem eliminandam, erit $4x^2 + 13x + 10 = 0$. Hac diviſa per 4, oritur æquatio *R*

$$R \quad x^2 + \frac{13x + 10}{4} = 0$$

Valor hujus x^2 ex hac æquatione *R* deſumptus *per Coroll. cit.* (hoc eſt $\frac{-13x - 10}{4}$) ſurrogetur in ſecunda æquatione *N* modo jam explicato, invenitur $\frac{49x + 98}{16} = 0$, & dividendo per $\frac{49}{16}$, habetur quotus $x + 2 = 0$, adeoque $x = -2$.

Ponatur

Ponatur demum hic valor in æquatione tertia *R*, provenit

$$4 - \frac{26 + 10}{4} = 0, \text{ hoc eſt } \frac{-26 + 26}{4} = 0$$

Proinde arguitur $x + 2 = 0$, unde iſta terminorum oritur oppoſitio, eſſe datarum æquationum diviſorem communem quæſitum.

SCHOL. *Hoc problema ſane ingenioſum, quod* Joh. Huddenius (*a*) Belga *vir ſubtiliſſimus excogitavit, uſum quoque habet ad inveniendum communem diviſorem duarum quarumcunque quantitatum. Nihil enim obſtat, quin illæ conſiderari poſſint inſtar duarum æquationum, & æquari nihilo: imo in illis ſumi poteſt ad libitum pro incognita quæcunque litera, quæ in utraque earum reperiatur. Sic in duabus ſequentibus quantitatibus, vel æquationibus* $d^3 - ad^2 + 2a^2b - 2abd = 0$ & $d^4 - b^2d^2 + a^2b^2 - a^2d^2 = 0$ *ſumi poteſt ad libitum pro incognita* d, a, *vel* b; & *procedendo eo ordine, quem nos in duobus exemplis ſequuti ſumus, invenietur communis earum diviſor* d — a, *ſiquidem* d *pro incognita fuerit aſſumpta.*

(*a*) V. in fine Geometr. Renati Descartes edit. 3. ann. 1683.

CAPUT VII.

De Reſolutione Æquationum compoſitarum, quæ radices rationales habent.

PROPOSITIO I.

Æquationis datæ radices rationales, ſi quæ ſint, invenire.

I. Inveniantur omnes ultimi termini diviſores *per Propoſ. 9. Cap. 1.* ex quibus una cum incognita fiant totidem æquationes ſimplices, & per illas ſigillatim dividatur æquatio: nam quæ exacte, & ſine ullo reſiduo æquationem datam dividet, erit radix rationalis quæſita.

Sit æquatio $x^3 - 9xx + 22x - 8 = 0$. Diviſores ultimi termini *per Prop. cit.* ſunt 1, 2, 4, 8. Quia vero ex mutatione ſignorum dignoſcitur *per num. 3. Prop. 1. Cap. 6.* radices omnes eſſe veras, proinde æquationes ſimplices, per quas tentari debet diviſio, erunt omnes cum ſigno —, nempe $x - 1 = 0$, $x - 2 = 0$, $x - 4 = 0$, $x - 8 = 0$. Diviſio autem fruſtra tentatur per $x - 1$, & per $x - 2$, quare nec $+1$, nec $+2$ poſſunt eſſe radices quæſitæ. Succedit autem diviſio ſine ullo reſiduo per $x - 4$, ut inſtituta diviſione patet.

$x - 4)$

$$
\begin{array}{r|l}
x-4) \quad x^3 - 9xx + 22x - 8 = 0 & x^2 \\
x^3 - 4xx \qquad\qquad\qquad & -5x \\
\hline
0 - 5xx + 22x \qquad\quad & +2 \\
- 5xx + 20x \qquad\quad & \\
\hline
0 + 2x - 8 & \\
+ 2x - 8 & \\
\hline
0 \quad 0 &
\end{array}
$$

Eſt ergo $+4$ una ex radicibus quæſitis, & ex hac diviſione oritur æquatio ſecundi gradus, nempe $xx - 5x + 2 = 0$. Cujus radices ut inveniantur, tentanda non eſt diviſio per $x - 1 = 0$, aut per $x - 2 = 0$. Nam cum per has quantitates tota æquatio ex dictis diviſibilis non fuerit, neque una ejusdem pars diviſibilis erit.

Tentetur itaque diviſio per $x - 4 = 0$, & per $x - 8 = 0$: ſed cum neque per hos diviſores diviſio exacte ſuccedat, ſignum eſt, alias duas radices non eſſe rationales. Quæ tamen obtineri poterunt *per Prop. 1. & 3. Cap. 8.*, eritque altera $= \frac{5}{2} + \frac{1}{2}\sqrt{17}$, per quam ſi dividatur eadem æquatio ſecundi gradus $xx - 5x + 2 = 0$, emerget tertia radix quæſita $= \frac{5}{2} - \frac{1}{2}\sqrt{17}$.

II. Sit alia æquatio $x^3 - 3x^2 - 10x + 24 = 0$, cujus radices rationales inquiruntur.

Diviſores ultimi termini *per Prop. 9. Cap. 1.* ſunt 1, 2, 3, 4, 6, 8, 12, 24. Ex ſignis autem æquationis propoſitæ *per num. 3. Prop. 1. Cap. 6.* patet radices fore duas quidem veras, & unam falſam, proinde æquationes ſimplices, per quas tentari debet diviſio, erunt tam cum

ſigno +, quam cum ſigno —, hoc eſt $x - 1 = 0$, $x + 1 = 0$, $x - 2 = 0$, $x + 2 = 0$, $x - 3 = 0$, $x + 3 = 0$ &c. Diviſio autem fruſtra tentatur per $x - 1$, & per $x + 1$. Sed exacte ſuccedit per $x - 2$, per $x + 3$, & per $x - 4$. Sunt ergo radices quæſitæ $+2$, -3, $+4$, ut computanti fit evidens.

III. Demum ſit æquatio $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$, cujus radices rationales quæruntur.

Ex ſignis æquationis apparet *per num. 3. Prop. 1. Cap. 6.* radices tres eſſe veras, & unam falſam. Inveniantur omnes diviſores ultimi termini *per Prop. 9. Cap. 1.* Cum diviſio exacte fieri non poſſit per $x - 1$, neque per $x + 1$, ſuccedat autem ſine ullo reſiduo per $x - 2$, erit $+2$ una ex radicibus veris.

Ex diviſione hujusmodi oritur æquatio $x^3 - 2x^2 - 23x + 60 = 0$, quæ fruſtra dividitur per $x - 2$, vel per $x + 2$, ſuccedit tamen exacta diviſio per $x - 3$. Habetur ergo $+3$ radix vera, & ex diviſione oritur æquatio ſecundi gradus $x^2 + 1x - 20 = 0$. Cujus unam radicem eſſe veram, alteram falſam ex ſignis ſatis apparet. Dividatur itaque per $x - 4$, diviſio ſuccedit ſine reſiduo, & quotus eſt $+5$, proinde aliæ duæ radices habentur $+4$, -5 una vera, altera falſa.

Schol. *Æquationis diviſionem per tot diviſores tentare ſi cui res moleſta ſit, ſequenti methodo uti poterit, quæ diviſores illos ad pauciores reducit. Ceterum ſi nullus ſit diviſor, qui datam æquationem exacte & ſine ullo reſiduo dividere poſſit, ſignum eſt, nullam dari radicem rationalem, quod notetur.*

PRO-

PROPOSITIO II.

Radices rationales propius inveſtigare.

I. Sit æquatio $x^3 - 9x^2 + 26x - 24 = 0$, cujus radices rationales quærimus. Inveniantur *per Prop. 9. Cap. 1.* diviſores ultimi termini, nempe 1, 2, 3, 4, 6, 8, 12, 24. Deinde æquationis datæ radices minuantur unitate *per Prop. 4. Cap. 6.* hoc eſt fiat $x - 1 = y$, ideoque erit $x = y + 1$, unde

$$\begin{array}{r|rrrr} x^3 & = y^3 & + 3y^2 & + 3y & + 1 \\ - 9x^2 & & - 9y^2 & - 18y & - 9 \\ + 26x & & & + 26y & + 26 \\ - 24 & & & & - 24 \\ \hline \text{Summa} & y^3 & - 6y^2 & + 11y & - 6 = 0 \end{array}$$

Diviſores ultimi termini hujus ſecundæ æquationis ſunt 1, 2, 3, 6. Cum autem radices hujus deficiant unitate a radicibus datæ æquationis, hæ, addita unitate, fient æquales prioribus, hoc eſt

$$\begin{array}{c} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ 6 + 1 = 7 \end{array}$$

Ex omnibus itaque diviſoribus ultimi termini primæ æquationis, nempe 24, illi tantum ſeligantur, qui cum his convenire, & utrique æquationi communes eſſe repe-
riuntur,

riuntur, ſcilicet 2, 3, 4, nam 7 non reperitur inter diviſores primæ æquationis. Ecce tibi tres tantum diviſores, ſeu melius tres radices (pauciores enim eſſe non poſſunt *per num. 1. Prop. 1. Cap. 6.*) datæ æquationis.

II. Quod ſi æquatio data radices habeat partim veras, partim falſas, diviſores ultimi termini æquationis transformatæ & augeri & minui debent ea quantitate, qua minuta fuit radix æquationis propoſitæ, ut detegantur radices tam veræ, quam falſæ. Exemplo res fit clariſſima.

Sit æquatio $x^4 * - 15x^2 + 10x + 24 = 0$, quæ duas habet radices veras $+2$, $+3$, & duas falſas -4, -1, quæ tamen ſupponuntur non cognitæ. Minuatur æquatio unitate *per Prop. 4. Cap. 6.* hoc eſt, fiat $x - 1 = y$, erit $x = y + 1$, proinde

$$\begin{array}{r|rrrrr} x^4 & = y^4 + 4y^3 + 6y^2 + & 4y + & 1 \\ -15x^2 & -15y^2 - & 30y - & 15 \\ +10x & + & 10y + & 10 \\ +24 & & + & 24 \\ \hline \text{Summa} & y^4 + 4y^3 - 9y^2 - & 16y + & 20 = 0 \end{array}$$

Radices veræ hujus æquationis deficiunt unitate a radicibus veris æquationis datæ *per Cor. 2. Prop. 4. Cap. 6.* Contra vero radices falſæ erunt unitate majores. Itaque diviſores ultimi termini 20 (inter quos quæſitæ radices latent) ſi augeantur unitate, fient æquales radicibus veris datæ æquationis. Contra vero ſi minuantur unitate, fient æquales radicibus falſis ipſius æquationis datæ. Diviſores ultimi termini ſunt 1, 2, 4, 5, 10, 20 *per Prop. 9. Cap. 1.* qui unitate aucti fiunt 2, 3, 5, 6, 11, 21. Conferantur

cum

cum diviſoribus ultimi termini æquationis datæ, hoc eſt 24; nempe cum 1, 2, 3, 4, 6, 8, 12, 24, tres tantum utrique æquationi communes inveniuntur, ſcilicet 2, 3, 6. Ex quibus duæ ſunt radices veræ quæſitæ, hoc eſt 2 & 3. Minuti vero unitate ſunt 0, 1, 3, 4, 9, 19, qui ſi conferantur cum diviſoribus æquationis datæ, nempe cum 1, 2, 3, 4, 6, 8, 12, 24, inveniuntur tres tantum cum his convenire, ſcilicet 1, 3, 4, ex quibus habentur duæ radices falſæ quæſitæ -1, -4.

COROL. *Redactis hoc pacto ad pauciſſimos ultimi termini diviſoribus, facile inveniuntur radices rationales, vel dividendo per illos datam æquationem, ut in propoſitione antecedente factum eſt; vel illos ſubſtituendo in æquatione propoſita loco incognitæ, ut docuimus* in Propoſ. 1. Cap. 6. num. 5. *Ratio autem hujus methodi fundatur* in Coroll. 1. & 2. Prop. 4. Cap. 6.

SCHOL. *Hujus inventi laudem* Jacobo a Waeſſenaer *Ultrajectino egregio Geometræ* Franciſcus a Schooten (*a*) *acceptam refert.*

PROPOSITIO III.

Idem problema in æquationibus literalibus.

Sit æquatio ex literis compoſita, cujus radices rationales inquiruntur,

$$\begin{aligned} x^3 - ax^2 &- abx + abd = 0 \\ + bx^2 &+ adx \\ - dx^2 &- bdx \end{aligned}$$

Z I. Cum

(*a*) Comment. in lib. III. Geometr. Carteſ. p. m. 3?

I. Cum ſingulæ radices appareant in ſecundo termino ſub ſigno contrario *per num.* 2. *Prop.* 1. *Cap.* 6. tentari debet diviſio per $x - a = 0$, aut per $x + b = 0$, quæ cum exacte, & ſine ullo reſiduo ſuccedat, patet radices datæ æquationis eſſe $+a$, $-b$, $+d$.

II. Quod ſi æquatio ſit magis compoſita, fiat æquatio ex ſingulis terminis, in quibus ſunt eædem literæ, & ſupponantur æquales zero *per Cor.* 2. *Prop.* 2. *Cap.* 6. ut ſic obtineri poſſit incognitæ valor. Exemplo res fit clarior. Sed data æquatio

$$\begin{array}{l} x^3 - 2ax^2 + a^2x - abb = 0 \\ \quad\;\; - bx^2 + abx - b^3 \\ \qquad\qquad + bbx \end{array}$$

Fiat æquatio ex omnibus terminis, in quibus reperitur quantitas bb, & ſupponatur æqualis zero, id eſt $bbx - abb - b^3 = 0$. Hæc dividatur per bb, quotus $x - a - b = 0$ eſt diviſor, per quem ſi dividas æquationem datam, nihil remanet, proinde $a + b$ eſt una ex illius radicibus; & ex diviſione oritur æquatio ſecundi gradus $x^2 - ax + bb = 0$ cujus radices facile erit invenire *per Prop.* 1. & 3. *Cap.* 8.

III. Sit demum æquatio, cujus radices rationales quæruntur

$$\begin{array}{l} x^4 - ax^3 + abx^2 - abcx + abcd = 0 \\ \quad\;\; - bx^3 + acx^2 - abdx \\ \quad\;\; - cx^3 + adx^2 - acdx \\ \quad\;\; - dx^3 + bcx^2 - bcdx \\ \qquad\quad\;\; + bdx^2 \\ \qquad\quad\;\; + cdx^2 \end{array}$$

Fiat æquatio ex omnibus illis terminis, in quibus reperitur quan-

quantitas abd, eos ſupponendo æquales zero, nempe $-abdx+abcd=0$, erit $abdx=abcd$, & $x=c$.

Dividatur æquatio propoſita per $x-c=0$, nihil remanet, proinde $+c$ eſt una ex illius radicibus. Diviſa autem ipſamet data æquatione per $x-c=0$, oritur æquatio tertii gradus, quæ ſequitur.

$$\begin{aligned} x^3 - ax^2 &+ abx - abd = 0 \\ - bx^2 &+ adx \\ - dx^2 &+ bdx \end{aligned}$$

Fiat iterum æquatio ex omnibus illis terminis, in quibus reperitur ad, eos ſupponendo æquales zero, hoc eſt $adx - abd = 0$, erit $x=b$; & cum diviſio, quæ tentatur per $x-b=0$, exacte ſuccedat, habetur ſecunda radix datæ æquationis, eademque ratione reliquæ duæ obtineri poſſunt.

PROPOSITIO IV.

Alia methodus æquationes compoſitas reſolvendi.

INveniendæ ſunt radices æquationis propoſitæ, nempe

$$\begin{aligned} x^4 - ax^3 &+ abx^2 - abcx + abcd = 0 \\ - bx^3 &+ acx^2 - abdx \\ - cx^3 &+ adx^2 - acdx \\ - dx^3 &+ bcx^2 - bcdx. \\ &+ bdx^2 \\ &+ cdx^2 \end{aligned}$$

Primo dividatur æquatio in duas æquationes, quarum altera contineat omnes terminos, in quibus quantitas c reperitur, altera reliquos, erit

$$\begin{array}{l} 1.^{a}\ -cx^3 + acx^2 - abcx + abcd = 0 \\ \qquad\quad + bcx^2 - acdx \\ \qquad\quad + cdx^2 - bcdx \end{array}$$

$$\begin{array}{l} 2.^{a}\quad x^4 - ax^3 + abx^2 - abdx = 0 \\ \qquad\ - bx^3 + adx^2 \\ \qquad\ - dx^3 + bdx^2 \end{array}$$

Patet, primam dividi poſſe per quantitatem c, ſecundam per x: quo facto, habetur

$$\begin{array}{l} 1.^{a}\quad -x^3 + ax^2 - abx + abd = 0 \\ \qquad\qquad + bx^2 - adx \\ \qquad\qquad + dx^2 - bdx \end{array}$$

$$\begin{array}{l} 2.^{a}\quad x^3 - ax^2 + abx - abd = 0 \\ \qquad\quad - bx^2 + adx \\ \qquad\quad - dx^2 + bdx \end{array}$$

Harum prima per ſecundam exacte dividitur, ut patet, & quotus eſt -1; unde apparet, haberi ex hac ſecunda æquatione diviſorem exactum primæ. Per hanc ſi æquatio propoſita dividatur, habebitur $x - c = 0$, una ſcilicet ex radicibus quæſitis. Reſtat autem æquatio ipſa ad gradum inferiorem depreſſa, ſcilicet

$$\begin{array}{l} x^3 - ax^2 + abx - abd = 0 \\ \quad\ - bx^2 + adx \\ \quad\ - dx^2 + bdx \end{array}$$

Hæc

Hæc ſimiliter dividatur in duas alias æquationes, in quarum una ſint termini, in quibus quantitas b reperitur, in alia reliqui, erit

$$1.^a \quad -bx^2 + abx - abd = 0 \\ \quad + adx$$

$$2.^a \quad x^3 - ax^2 + adx = 0 \\ \quad - dx^2$$

Dividatur prima per b, ſecunda per x; erit

$$1.^a \quad -x^2 + ax - ad = 0 \\ \quad + dx$$

$$2.^a \quad x^2 - ax + ad = 0 \\ \quad - dx$$

Quarum prima per ſecundam exacte dividitur: nam quotus eſt -1, & nihil remanet. Per hanc ſecundam igitur dividatur ſuperior tertii gradus, habebitur $x - b$ pro ſecunda vera radice datæ æquationis, & remanet æquatio ſecundi gradus, videlicet

$$x^2 - ax + ad = 0 \\ - dx$$

Quæ ſimiliter diviſibilis eſt in alias duas, nimirum

$$-ax + ad = 0, \ \& \ x^2 - dx = 0$$

Harum una diviſa per a, altera per x, habentur $-x + d = 0$, & $x - d = 0$; ex quibus prima cum ſit exacte di-

viſibilis per ſecundam (nam quotus eſt -1, & nihil remanet) dividi per hanc poterit æquatio ſuperior ſecundi gradus, nempe

$$\begin{aligned} x^2 - ax &+ ad = 0 \\ - dx & \end{aligned}$$

atque erit $x - d$ tertia radix quæſita: quarta autem $x - a$, facta diviſione, habetur ex quoto.

COROLL. I. *Vel ex hoc uno exemplo apparet, methodum in eo eſſe: 1.° Ut æquatio propoſita in duas dividatur æquationes, quarum quælibet contineat terminos iiſdem fere literis conflatos. 2.° Ut dividendo per quantitatem, quæ in illis communiter reperitur, earum æquationum terminos, ad inferiorem gradum æquationes illæ deprimantur, 3.° Ut obſervetur, an una æquatio per alteram ſit exacte & ſine ullo reſiduo diviſibilis: nam tunc illa erit communis menſura, hoc eſt dividere poterit non ſolum illam æquationis partem, ſed etiam totam æquationem datam, & aliquam ex illius radicibus exhibere. Ratio eſt, quia quantitas, quæ exacte dividit partes, etiam totum dividat neceſſe eſt. Ceterum æquatio propoſita dividi poterat in duas alias alio, vel alio modo, ac a nobis factum eſt, ut conſideranti patebit.*

COROLL. II. *Quod ſi non omnes æquationis datæ radices hac ratione inveniri poſſint, indicio eſt, reliquas eſſe irrationales, de quibus inferius.*

PRO-

PROPOSITIO V.

Æquationes compoſitas, in quibus duæ, vel plures radices ſunt æquales, reſolvere.

I. SI data æquatio duas æquales radices habere ſupponitur, multiplicetur per quancunque progreſſionem Arithmeticam, hoc eſt primus terminus æquationis per primum terminum progreſſionis, ſecundus terminus per ſecundum &c. & productum, quod inde fit, erit $= 0$, ſicuti æquatio ipſa data ſupponitur $= 0$. Deinde inveniatur æquationum duarum, datæ & productæ, communis diviſor *per Prop.* 13. *Cap.* 6. per quem diviſa ipſamet æquatio data, quoties opus ſit, exhibebit radices æquales. Sit æquatio

$$\begin{array}{cccc} x^3 - & 5x^2 + & 8x - & 4 = 0 \\ 3 & 2 & 1 & 0 \\ \hline 3x^3 - & 10x^2 + & 8x - & 0 = 0 \end{array}$$

Hæc diviſa per x fit uno gradu inferior, nempe

$$3x^2 - 10x + 8 = 0$$

Diviſor autem communis hujus & datæ æquationis eſt $x - 2 = 0$ *per Prop.* 13. *Cap.* 6. per quem data æquatio bis diviſa dat radices 2, 2, 1.

II. Poterat autem eadem æquatio per progreſſionem Arithmeticam alio, vel alio modo multiplicari, ſcilicet

$$\begin{array}{cccc} x^3 - & 5x^2 + & 8x - & 4 = 0 \\ 0 & 1 & 2 & 3 \\ \hline 0 - & 5x^2 + & 16x - & 12 = 0 \end{array}$$

Hujus

Hujus enim & propoſitæ æquationis communis diviſor eſt ut antea, $x - 2 = 0$.

III. Similiter ſi ſecundum terminum auferre libeat, ita progreſſio Arithmetica diſponitur.

$$\begin{array}{cccc} x^3 - 5x^2 & + 8x & - 4 & = 0 \\ 1 \quad 0 & -1 & -2 & \\ \hline x^3 \quad * & - 8x & + 8 & = 0 \end{array}$$

Cujus quidem æquationis & datæ communis diviſor ſemper reperitur $x - 2 = 0$.

IV. Si æquatio data tres habeat radices æquales, bis multiplicentur ejus termini per terminos progreſſionis Arithmeticæ, ut ſupra: ſi habeat quatuor radices æquales, ter multiplicentur: ſi quinque radices æquales, quater &c. Sit æquatio, quæ tres radices æquales habere ſupponitur.

$$\begin{array}{ccccc} x^4 & * & - 6x^2 & + 8x & - 3 = 0 \\ 0 & 1 & 2 & 3 & 4 \\ \hline 0 & * & - 12x^2 & + 24x & - 12 = 0 \\ & & 0 & 1 & 2 \\ \hline & & 0 & + 24x & - 24 = 0 \end{array}$$

Communis diviſor eſt $x - 1$; per quem ſi ter dividatur æquatio propoſita, quotus dabit tres radices æquales $+1$, $+1$, $+1$. Quarta vero (cum in æquatione ſecundus terminus deſit) erit -3 *per Prop.* 1. *Cap.* 6. *num.* 4.

Demonſtr. Per progreſſionem Arithmeticam æquatio data in aliam convertitur, quæ continet eaſdem radices æquales una minus. Nam ut ſuperiora exempla docent, ea potiſſi-

tiſſimum progreſſio aſſumitur, quæ vel incipit a zero, vel deſinit in zerum, ut æquatio, quæ inde producitur, fiat uno gradu inferior, ideoque radicum numerus decreſcit, fitque una radice minor. Per iteratas autem ejuſmodi multiplicationes ad eam æquationem devenitur, in qua una tantum æqualium radicum continetur: proinde ſi hujus & propoſitæ æquationis communis diviſor inveniatur, ille æqualium radicum unam dabit.

COROLL. I. *Manifeſtum eſt, hac ratione tolli poſſe ex æquatione data terminum quem quis voluerit, collocando ſub eo zerum, ut allata exempla docent.*

COROLL. II. *Si communis diviſor nullo modo reperiri poſſit, tunc infertur, radices æquales eſſe irrationales, quæ non ſunt hujus loci.*

SCHOL. I. *Communis diviſor non eſt neceſſe, ut inveniatur inter æquationem datam & æquationem ex Arithmetica progreſſione productam, ut in primo exemplo factum eſt: ſed commodius aliquando ſumitur inter duas æquationes productas, ut inter* $3x^2 - 10x + 8 = 0$, *&* $-5x^2 + 16x - 12 = 0$, *quæ habentur in eodem primo exemplo, quarum diviſor communis eſt* $x - 2$, *ut prius.*

SCHOL. II. *Pulcherrimum hoc problema, quod* Johanni Huddenio (a) *Belgæ acerrimi ingenii viro debemus, quanti ſit uſus in Geometria ſublimiori pro ducendis Tangentibus, pro determinandis quæſtionibus de Maximis, & Minimis aliiſque pluribus, ſatis intelliget, qui ad eam gradum faciet.*

(a) De reductione æquat. Reg. x. in fine Geom. Carteſ. pag. m. 433.

CAPUT VIII.

De Æquationibus Quadraticis.

SI æquationum radices inveniri nequeant per ea, quæ in Capite præcedenti explicavimus, ſignum erit, radices irrationales, vel etiam imaginarias in illis contineri, aliaque via reſolutionem illarum eſſe ineundam. Proinde regulas peculiares pro æquationibus ejuſmodi trademus, a *quadraticis* incipiendo; quæ ſi fractionibus, aut radicalibus implicentur, prius ab illis expediri debent per ea, quæ docuimus *in Prop.* 2. 8. *&* 9. *Cap.* 6.

PROPOSITIO I.

Æquationes ſecundi gradus, ſeu quadraticas reſolvere.

I. SI æquatio *quadratica* ſit pura, ut $x^2 = ab$, extracta hinc inde ſecunda radice, habetur valor quæſitus, nempe $x = \sqrt{ab}$. Eadem ratione $x^2 = 14400$; extracta pariter radice, erit $x = \sqrt{14400}$; & ſi per communem Arithmeticam actu radix extrahatur, habetur $x = 120$.

II. Si æquatio fuerit affecta, ut $x^2 + ax = b^2$, regula hæc erit: ex dimidio coefficientis ſecundi termini, nempe ex $\frac{1}{2}a$, fiat quadratum $\frac{1}{4}aa$, quod utrique membro æquationis addatur, ut potentia fiat completa, unde radix ſecunda extrahi poſſit. Ecce exemplum. Sit æquatio data

x^2

$$
\begin{array}{ll}
 & x^2 + ax = bb \\
\textit{Adde} & \quad \tfrac{1}{4}a^2 \quad \tfrac{1}{4}a^2 \\
\hline
 & x^2 + ax + \tfrac{1}{4}a^2 = bb + \tfrac{1}{4}a^2 \\
\textit{Extr. rad.} & x + \tfrac{1}{2}a = \sqrt{bb + \tfrac{1}{4}a^2} \\
 & x = \sqrt{bb + \tfrac{1}{4}a^2} - \tfrac{1}{2}a
\end{array}
$$

Sit aliud exemplum $x^2 - 3ax = b^2$. Fiat ex dimidio coefficientis $\frac{3}{2}a$ quadratum $\frac{9}{4}a^2$, quod utrinque addatur, erit

$$
\begin{array}{ll}
 & x^2 - 3ax = b^2 \\
\textit{Adde} & \quad \tfrac{9}{4}a^2 \quad \tfrac{9}{4}a^2 \\
\hline
 & x^2 - 3ax + \tfrac{9}{4}a^2 = b^2 + \tfrac{9}{4}a^2 \\
\textit{Extra. rad.} & x - \tfrac{3}{2}a = \sqrt{b^2 + \tfrac{9}{4}a^2} \\
 & x = \sqrt{b^2 + \tfrac{9}{4}a^2} + \tfrac{3}{2}a
\end{array}
$$

Demonſtratio ex ipſa potentiæ ſecundæ geneſi emanat. Nam in binomio ad potentiam ſecundam elevato quantitas ſecundi termini eſt duplum facti ex utraque binomiæ radicis parte. Cum igitur habeatur (in primo exemplo) x pars una, erit $\frac{1}{2}a$ pars altera radicis, ex qua potentia ſecunda completur.

COROLL. *Hinc patet, in ejuſmodi æquationibus radicem ſecundam obtineri, accipiendo ſummam, vel differentiam radicum primi & tertii termini potentiæ completæ. Accipitur ſumma, cum omnes termini ſunt poſitivi, ut* $x + \frac{1}{2}a$ *in primo exemplo; differentia vero, ſi ſecundus terminus ſit negativus, ut* $x - \frac{3}{2}a$ *in ſecundo exemplo.*

Schol. I. *Æquationes affectæ ſecundi gradus reſolvuntur etiam facillime, ſublato ſecundo termino* per Prop. 5, Cap. 6. *Sit enim eadem æquatio reſolvenda* $x^2 - 3ax = b^2$, *ſeu* $x^2 - 3ax - b^2 = 0$. *Ut tollatur ſecundus terminus, fiat* $x = y + \frac{3}{2}a$, *erit*

$$\begin{array}{r|rrrr} x^2 & = y^2 & + 3ay & + \frac{9}{4}a^2 & \\ -3ax & \cdot & - 3ay & - \frac{9}{2}a^2 & \\ -b^2 & & & - b^2 & \\ \hline & y^2 & * & - \frac{9}{4}a^2 & - b^2 = 0 \end{array}$$

Eſt ergo $y^2 = \frac{9}{4}a^2 + b^2$, *& extracta radice*, $y = \sqrt{\frac{9}{4}a^2 + b^2}$. *Ponatur hic valor loco* y *in æquatione* $x = y + \frac{3}{2}a$, *erit* $x = \frac{3}{2}a + \sqrt{\frac{9}{4}a^2 + b^2}$ *omnino ut prius.*

Schol. II. *Omnis æquatio quadratica duas habet radices, alteram affirmativam, negativam alteram. Nam quadratum quodlibet, v.g.* 25, *tam oritur ex* 5×5, *quam ex* -5×-5. *Proinde ſi fuerit æquatio* $x^2 = ab$, *valor erit* $x = \pm\sqrt{ab}$, *hoc eſt valor ipſius* x *tam habetur per radicem poſitivam* $+\sqrt{ab}$, *quam per negativam* $-\sqrt{ab}$. *Sic etiam in exemplo ſuperiore* Schol. 1. *radix poſitiva eſt* $x = \frac{3}{2}a + \sqrt{\frac{9}{4}a^2 + b^2}$, *negativa vero* $x = \frac{3}{2}a - \sqrt{\frac{9}{4}a^2 + b^2}$. *Quod adeo certum eſt, ut ſi alterutra ponatur in æquatione data loco* x, *termini omnes evaneſcunt, proinde utraque eſt vera radix* per num. 5. Prop. 1. Cap. 6. *Brevitatis autem gratia utrunque ſignum apponunt, unum ſub alio* $\pm$. *Quod pro iis, quæ dicenda ſunt, notetur.*

PRO-

PROPOSITIO II.

Æquationes ſecundi gradus alia ratione expenduntur.

Sit æquatio generalis $x^2 + 2px + q = 0$, repræſentans omnes æquationes ſecundi gradus, ita ut $+ 2p$ repræſentet coefficientem ſecundi termini cum ſuo ſigno, & $+ q$ ultimum terminum cum ſuo pariter ſigno, ut explicavimus *in Coroll.* 1. *Prop.* 2. *Cap.* 6. Summa radicum ejuſdem æquationis ſit $= 2f$, earumque differentia $= 2g$, erit radix major $f + g$, minor vero $f - g$ per *Theor.* 3. *Prop.* 3. *Cap.* 5. Cumque utraque ſit unus valor ipſius incognitæ x, erit $x = f + g$, & $x = f - g$. Fiant igitur (mutatis ſignis) æquationes ſimplices $x - f - g = 0$, & $x - f + g = 0$, ex quarum multiplicatione oritur æquatio

$$x^2 - 2fx + ff = 0 \\ - gg$$

Quæ ex ſuppoſitione æqualitatis radicum erit æqualis æquationi $x^2 + 2px + q = 0$. Fiat ergo terminorum comparatio (neglecto primo utriuſque termino) erit $- 2f = 2p$, unde habetur $f = - p$, & $ff = p^2$. Item $ff - gg = q$, ſeu $gg = ff - q$, & ponendo p^2 loco ff, erit $gg = p^2 - q$, extractaque radice, habetur $g = \sqrt{p^2 - q}$.

Determinato ita valore ipſarum f & g, erit radix major $f + g = - p + \sqrt{p^2 - q}$, minor vero $f - g = - p - \sqrt{p^2 - q}$.

Coroll. *Hinc habentur quatuor illæ formulæ generales pro æquatione quacunque ſecundi gradus reſolvenda, de quibus in ſequenti Propoſitione.*

PROPOSITIO III.

Æquationes ſecundi gradus per formulas generales reſolvere.

I. ÆQuationes ſecundi gradus affectæ quatuor modis ratione ſignorum variari poſſunt, ideoque quatuor formulis generalibus exprimi ſolent, in quibus $2p$ ſemper denotat quantitatem cognitam ſecundi termini, q vero quantitatem cognitam tertii, videlicet

$$\begin{array}{rl} \text{I.} & x^2 - 2px - q = 0 \\ \text{II.} & x^2 + 2px - q = 0 \\ \text{III.} & x^2 + 2px + q = 0 \\ \text{IV.} & x^2 - 2px + q = 0 \end{array}$$

II. Reperto valore incognitæ x *per Prop. antec.* habentur ſequentes formulæ generales cum duplici ſigno $\pm$ ob duplicem radicem affirmativam & negativam ex dictis *in Schol. 2. Prop. 1. hujus.*

$$\begin{array}{rl} \text{I.} & x = p \pm \sqrt{p^2 + q} \\ \text{II.} & x = -p \pm \sqrt{p^2 + q} \\ \text{III.} & x = -p \pm \sqrt{p^2 - q} \\ \text{IV.} & x = p \pm \sqrt{p^2 - q} \end{array}$$

1. Pro-

1. Propoſita ſit igitur æquatio ſecundi gradus reſolvenda $x^2 - 10x - 6 = 0$.

Ex ſignis ſtatim apparet, hanc æquationem pertinere ad primam formulam, $x^2 - 2px - q = 0$. Comparentur mutuo termini, erit $2p = 10$, & $p = 5$. Item $q = 6$. Eſt autem radix primæ formulæ generalis $x = p + \sqrt{p^2 + q}$, in qua ſi ponantur loco p & q earum valores, erit $x = 5 + \sqrt{25 + 6}$, ſeu $x = 5 + \sqrt{31}$. Altera ejuſdem formulæ radix $x = p - \sqrt{p^2 + q}$, hoc eſt $x = 5 - \sqrt{31}$.

2. Sit æquatio $x^2 + 1x - 2 = 0$, quæ ut ex ſignis liquet, pertinet ad ſecundam formulam $x^2 + 2px - q = 0$. Fiat comparatio terminorum, erit $2p = 1$, unde $p = \frac{1}{2}$, & $q = 2$. Radix formulæ ſecundæ eſt $x = -p + \sqrt{p^2 + q}$, ideoque poſitis loco p & q earum valoribus, habetur $x = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2}$, ſeu $x = -\frac{1}{2} + \sqrt{\frac{9}{4}} = 1$. Altera radix ejuſdem formulæ eſt $x = -p - \sqrt{p^2 + q}$, proinde $x = -\frac{1}{2} - \sqrt{\frac{9}{4}} = -2$.

3. Sit æquatio $x^2 + 4x + 6 = 0$ pertinens ad tertiam formulam, ut ex ſignis patet, $x^2 + 2px + q = 0$. Comparando terminos utriuſque, erit $2p = 4$ & $p = 2$, item $q = 6$. Radix autem generalis ejuſdem formulæ eſt $x = -p + \sqrt{p^2 - q}$, ſubſtitutis in eâ valoribus p & q, erit $= -2 + \sqrt{4 - 6}$, ſeu $x = -2 + \sqrt{-2}$. Altera radix erit $x = -2 - \sqrt{-2}$. Hinc apparet propoſitæ æquationis radices eſſe imaginarias.

4. Sit

4. Sit æquatio $x^2 - 6x + 8 = 0$, quæ ad quartam formulam ſpectat propter ſignorum ſimilitudinem, nempe ad $x^2 - 2px + q = 0$. Facta terminorum comparatione, erit $2p = 6$, & $p = 3$, item $q = 8$, quibus valoribus p & q poſitis in formula radicis generalis $x = p + \sqrt{p^2 - q}$, erit una radix $x = 3 + \sqrt{9 - 8}$, ſeu $x = 3 + \sqrt{1}$, altera vero $x = 3 - \sqrt{1}$; vel una $x = 4$, altera $x = 2$: ambæ ſunt poſitivæ.

Coroll. *Contingit non ſemel, ut tertiæ & quartæ formulæ radices ſint imaginariæ, cum ſcilicet* q *major eſt, quam* p^2, *ut ſupra in tertio exemplo, quod ſolutioni problematum nihil obſtat. Idem quoque accidit, ſi in æquatione deſit ſecundus terminus, tertius autem ſit cum ſigno* +, *ut* $x^2 + q = 0$. *Nam una radix erit* $x = -\sqrt{-q}$, *altera* $x = +\sqrt{-q}$, *proinde ambæ imaginariæ. Ceterum ubi ultimus æquationis terminus ſigno negativo afficitur, radices ſemper ſunt reales, ut ex prima & ſecunda formula apparet.*

PROPOSITIO IV.

Æquationes derivativas ſecundi gradus reſolvere.

I. ÆQuationes tribus tantum terminis conſtantes, quarum primus ad quatuor, ad ſex, & ad altiores quoque dimenſiones aſcendit, & in quibus exponentes terminorum incognitorum habent inter ſe & zero eandem differentiam, hoc eſt ſunt in eadem proportione Arithmetica, ut 4, 2, 0, vel 6, 3, 0 &c. dicuntur æquationes *deriva-*

rivativæ ſecundi gradus, & reſolvuntur ſimiliter ac ipſæmet ſecundi gradus æquationes, ut quæ afferuntur exempla docent.

Sit æquatio $x^4 - 6x^2 - 4 = 0$. Pone $x^2 = y$, erit $x^4 = y^2$, unde oritur $y^2 - 6y - 4 = 0$, quæ ſi reſolvatur per primam formulam generalem *Propoſ. antec.* habetur $y = 3 + \sqrt{13}$.

Quia vero poſita fuit $x^2 = y$, erit $x = \sqrt{y}$, adeoque $x = \sqrt{3 + \sqrt{13}}$.

II. Sit æquatio $x^6 + 4x^3 - 12 = 0$, pone $x^3 = y$, erit $x^6 = y^2$, unde oritur $y^2 + 4y - 12 = 0$. Quæ ſi reſolvatur per formulas generales *Prop. ant.*, cum ratione ſignorum pertineat ad ſecundam formulam, erit valor ipſius $y = -2 + \sqrt{16}$, ſeu $y = 2$. At vero cum poſita fuerit $x^3 = y$, erit $x = \sqrt[3]{y}$, hoc eſt $x = \sqrt[3]{2}$.

Hæc omnia ex ſolutione problematum, quæ ſequuntur, clarius elucefcent.

PROBLEMA I.

Invenire quadratum, cui ſi addatur ejus radix, fiat æquale unitati.

Sit quæſiti quadrati latus $= x$, erit per conditionem problematis æquatio

$$xx + x = 1$$

per Prop. 1. adde $\frac{1}{4} \quad \frac{1}{4}$

$$xx + x + \tfrac{1}{4} = \tfrac{5}{4}$$

 Extr.

Extr. radix $x+\frac{1}{2}=\sqrt{\frac{5}{4}}$

$$x=\sqrt{\tfrac{5}{4}}-\tfrac{1}{2}$$

$$xx=\tfrac{5}{4}-\sqrt{\tfrac{5}{4}}+\tfrac{1}{4}$$

adde $x=\;+\sqrt{\frac{5}{4}}-\frac{1}{2}$

$$xx+x=\tfrac{5}{4}-\tfrac{1}{4}=\tfrac{4}{4}=1$$

PROBLEMA II.

Duos numeros invenire, quorum ſumma a ſumma quadratorum, quæ ex ipſis fiunt, ſubtracta, relinquit 78. *Addita vero ad eorum factum, efficit* 39.

Sit numerorum quæſitorum ſumma $=2x$, eorumque differentia $=2y$, erit major $=x+y$, minor $=x-y$ *per Coroll. Theor. 3. Cap. 5.* Sit vero $b=39$, erunt $2b=78$.

Si fiant quadrata, & ex eorum ſumma $2x^2+2y^2$ auferatur ſumma numerorum $2x$, erit per primam problematis conditionem æquatio

Div. per 2) $2x^2+2y^2-2x=2b$

$$x^2+y^2-x=b$$

Quod ſi ad factum eorundem numerorum x^2-y^2 addatur eorum ſumma $2x$, oritur ex ſecunda problematis conditione æquatio altera

$$x^2-y^2+2x=b$$

Prop. 2. Cap. 5. $x^2+y^2-x=b$

$$2x^2+1x=2b$$

Div. per 2) $x^2+\frac{1}{2}x-b=0$

Hæc

Hæc autem finalis æquatio pertinet ad ſecundam formulam generalem *Propoſ. 3. hujus*, ex qua habetur radix $x = -p + \sqrt{p^2 + q} = -\frac{1}{4} + \sqrt{\frac{1}{16} + 39} = -\frac{1}{4} + \frac{25}{4} = 6$. Nam $p = \frac{1}{4}$, & $q = b = 39$.

Valor autem ipſius y innoteſcit ex prima æquatione $x^2 + y^2 - x = b$, ex qua $y^2 = b - x^2 + x$; proinde ſubſtituto valore modo invento pro x & x^2, habetur $y^2 = 39 - 36 + 6 = 9$, unde $y = 3$ & $x + y = 9$, $x - y = 3$.

SCHOL. Clavius (*a*) *putavit hoc problema, quod ipſe ænigma vocat, difficillimum; pro cujus ſolutione ipſemet Geometriæ opem implorat, contenditque illud ſine Geometria vix, aut nullo modo ſolvi poſſe. Quod quidem de Algebra* Numeroſa, *quæ id temporis obtinebat, fortaſſe verum. Ceterum quam facile nunc* per Speciosſam *a ſummis viris* Vieta & Carteſio *inductam hujusmodi problema ſine Geometria reſolvatur, jam vidimus. Atque hinc apparet quanta ſit hujus ſupra illam utilitas & præſtantia.*

PROBLEMA III.

Invenire tres numeros in proportione Arithmetica, quorum primus ſi multiplicetur per 1, *ſecundus per* 2, *& tertius per* 3, *producta ſint æqualia numero dato* 30. *Quadrata vero quæ ex ipſis numeris fiunt, æqualia ſint numero* 66.

Pone numerum datum $30 = a$, $66 = b$, & tres numeri quæſiti vocentur A, B, C. Sit primus quæſitorum

 $A = x$,

(*a*) V. Algeb. Cap. XXXI. Ænigma 58. p. m. 343.

$A=x$, tertius $C=y$, eorum ſumma $x+y$ ſi bifariam dividatur, dabit *per Coroll. Theor. 1. Cap. 5.* numerum ſecundum $B=\frac{1}{2}x+\frac{1}{2}y$.

Sunt autem ex prima problematis conditione

$$\begin{array}{llcl} A & x \times 1 & = & \\ B & \frac{1}{2}x+\frac{1}{2}y \times 2 & = & x+y \\ C & y \times 3 & = & 3y \\ \hline A+B+C & = 2x+4y=a & & \end{array}$$

Proinde

$$\begin{aligned} 2x &= a - 4y \\ x &= \tfrac{1}{2}a - 2y \; (=A) \\ \tfrac{1}{2}x &= \tfrac{1}{4}a - y \end{aligned}$$

Poſitoque hoc valore in quantitate $\frac{1}{2}x+\frac{1}{2}y\;(=B)$ erit $B=\frac{1}{4}a-\frac{1}{2}y$.

Sunt ergo tres numeri quæſiti

$$\begin{aligned} A &= \tfrac{1}{2}a - 2y \\ B &= \tfrac{1}{4}a - \tfrac{1}{2}y \\ C &= y \;(\text{nam } 2B - A = C = y) \end{aligned}$$

Eorumque quadrata erunt

$$\begin{array}{lcl} AA & = & 4y^2 - 2ay + \frac{1}{4}aa \\ BB & = & \frac{1}{4}y^2 - \frac{1}{4}ay + \frac{1}{16}aa \\ CC & = & yy \\ \hline \text{Summa} & = & 5\frac{1}{4}y^2 - 2\frac{1}{4}ay + \frac{5}{16}aa = b \\ & & 21y^2 - 9ay + 1\frac{1}{4}aa = 4b \\ \textit{Divid. per } 21 & & y^2 - \frac{3}{7}ay + \frac{5}{84}aa = \frac{4}{21}b \end{array}$$

Et

Et ſubſtitutis loco a & b numeris datis, erit

$$y^2 - 12\tfrac{6}{7}y + 53\tfrac{4}{7} = 12\tfrac{4}{7}$$

$$\textit{Subtr.} \quad 53\tfrac{4}{7} \quad 53\tfrac{4}{7}$$

$$y^2 - 12\tfrac{6}{7}y = -41$$

Hinc *per Prop. 1. vel 3. hujus* invenietur $y = 7$, proinde innoteſcent reliqui

$$A = \tfrac{1}{2}a - 2y = 1$$
$$B = \tfrac{1}{4}a - \tfrac{1}{2}y = 4$$
$$C = y = 7$$

PROBLEMA IV.

Datum numerum 26 in tres partes continuo proportionales Geometrice dividere, ut quadratum mediæ æquetur duplo facti mediæ & minimæ, & præterea ſextuplo ejuſdem minimæ.

Pone numerum datum dividendum $26 = a$, & ſit trium proportionalium minima $= x$, media $= y$, erit maxima $= a - x - y$. Cum autem factum maximæ & minimæ ſit æquale quadrato mediæ *per Theor. 5. Cap. 5.* erit

$$ax - xx - xy = y^2$$

Sed ex conditione problematis $y^2 = 2xy + 6x$, ergo

$$ax - xx - xy = 2xy + 6x$$

$$\textit{Divid. per } x \qquad a - x - y = 2y + 6$$

Hinc habentur tres partes proportionales

$$\begin{array}{rl} & 2y + 6 \\ & + y \\ & + x \\ \hline \text{Summa} & 3y + x + 6 = a \\ & 3y = a - x - 6 \\ \textit{Divid. per 3.} & y = \tfrac{1}{3}a - \tfrac{1}{3}x - 2 \end{array}$$

Ponătur brevitatis cauſa loco a numerus datus 26, erit

$$y = -\tfrac{1}{3}x + 6\tfrac{2}{3}$$
$$y^2 = \tfrac{1}{9}x^2 - 4\tfrac{4}{9}x + 44\tfrac{4}{9}$$

Cui æqualis ſupponitur $2yx + 6x$. Subſtituto proinde in hac quantitate pro $2y$ ejus valore, oritur æquatio

$$\begin{array}{rl} & -\tfrac{2}{3}x^2 + 19\tfrac{1}{3}x = \tfrac{1}{9}x^2 - 4\tfrac{4}{9}x + 44\tfrac{4}{9} \\ \textit{Subtr.} & \tfrac{1}{9}x^2 - 4\tfrac{4}{9}x \quad \tfrac{1}{9}x^2 - 4\tfrac{4}{9}x \\ \hline & -\tfrac{7}{9}x^2 + 23\tfrac{7}{9}x = 44\tfrac{4}{9} \\ \textit{Mult. per 9} & -7x^2 + 214x = 400 \\ \textit{Divid. per 7} & x^2 - 30\tfrac{4}{7}x = -57\tfrac{1}{7} \end{array}$$

Hinc *per Propoſ. 1. vel 3. hujus* innoteſcet $x = 2$, y $(= -\tfrac{1}{3}x + 6\tfrac{2}{3}) = 6$, & pars maxima $= 18$. Quæ ſimul conficiunt numerum 26.

PRO

PROBLEMA V.

Datis duobus numeris, duos alios invenire, inter quos datorum alter ſit Geometrice, alter Arithmetice proportionalis.

Sit ex datis numeris alter $= a$, alter $= b$: quæſitorum primus $= x$, ſecundus $= y$, erit per primam problematis conditionem

$$x \,.\, a :: a \,.\, y$$

Theor. 4. Cap. 5. $\quad xy = aa$, & $x = \frac{aa}{y}$

Similiter per ſecundam problematis conditionem erit

$$x \,.\, b \because b \,.\, y$$

Theor. 1. Cap. cit. $\quad x + y = 2b$, hinc $x = 2b - y$.

Et (*ob* $x = \frac{aa}{y}$) $\frac{aa}{y} = 2b - y$

Multiplic. per y $\quad aa = 2by - yy$

$$yy - 2by = -aa$$

Prop. 1. hujus $\quad$ *adde* $\quad b^2 \quad b^2$

$$yy - 2by + b^2 = b^2 - aa$$

Extr. rad. $\quad y - b = \sqrt{b^2 - a^2}$

$$y = b + \sqrt{b^2 - a^2}$$

$$x\,(= 2b - y) = b - \sqrt{b^2 - a^2}$$

Sit $a = 3$, $b = 5$, erit $y = 5 + \sqrt{25 - 9} = 5 + 4 = 9$, & $x = 5 - \sqrt{25 - 9} = 5 - 4 = 1$. Inventi ſunt ergo 1

&

& 9, inter quos datorum alter 3 mediat Geometrice, alter 5 Arithmetice.

Sit $a=4$, $b=5$, erit $y=8$, $x=2$, habentur ergo 2 & 8, inter quos unus datorum 4 eſt medius Geometrice proportionalis, alter 5 eſt medius Arithmetice.

PROBLEMA VI.

Duo mercatores ſocietatem ineunt. Primus ſummam neſcio quam poſuit, & manſit in ſocietate menſes 12. *Alter poſuit aureos* 30, *quos in ſocietate reliquit menſibus* 17. *Invenerunt lucrum eſſe aureorum* $18\frac{3}{4}$. *Primus pro pecunia in ſortem collata una cum lucro habuit aureos* 26. *Quæritur quantum ipſemet primus poſuerit.*

Eſto ſumma a primo poſita $=x$, tempus, ſeu menſes $12=a$: ſumma alterius $30=b$, menſes $17=c$; erit ſumma primi cum ſuo tempore $=ax$, ſumma alterius cum tempore $=bc$, & utriusque ſumma $=ax+bc$: lucrum ex ſocietate factum ſit $=d$. Sors autem primi cum lucro eſt $26=f$. Ut innoteſcat lucrum primi, fiat

Cor. 1. Theor. 4. Cap. 4. $ax+bc \,.\, d :: ax \,.\, \frac{adx}{ax+bc}$

Cui ſi addatur pecunia ab ipſo in ſortem collata, nempe x, habetur æquatio

$$x+\frac{adx}{ax+bc}=f.$$

Et

Et ad tollendas fractiones multiplicando per $ax + bc$, habetur

$$axx + bcx + adx = afx + bcf$$

Propoſ. 2. *Cap.* 6.
$$axx + bcx = bcf + adx - afx$$

Poſitis autem numeris determinatis loco a, b, c & f, erit

$$12x^2 + 423x = 13260$$

Divid. per 12 $x^2 + 35\frac{1}{4}x = 1105$

Hinc *per Prop.* 1. *vel* 3. *bujus* invenitur $x = 20$ pro pecunia a mercatore primo poſita.

SCHOL. *Hoc problema fuit Florentiæ publice propoſitum an.* 1653. *Cumque diu inſolutum remanſiſſet, ubi ad notitiam* Caroli Rinaldini *illud venit, fuit ab eodem ſtatim reſolutum, ut ipſe teſtatur in Arte Analytica Mathematum pag.* 526. *Expedita tamen magis hæc noſtra ſolutio eſt.*

Problemata Geometrica.

PROBLEMA I.

Datis in triangulo rectangulo perpendiculari AB, *&* *aggregato hypothenuſæ ac baſis* AC+BC, *hypothenuſam & reliquum latus invenire.* Fig. 1.

Sit perpendicularis $AB = a$, aggregatum hypothenuſæ ac baſis $AC + BC = b$, & $AC = x$, erit $BC = b - x$, proinde *per* 47. *l.* 1. *Eucl.*

$$x^2 = b^2 - 2bx + x^2 + a^2$$
$$b^2 - 2bx + a^2$$
$$2bx = a^2 + b^2$$
$$x = \frac{a^2 + b^2}{2b} = AC$$

$$BC(= b - x) = \frac{b^2 - a^2}{2b}$$

Schol. *Ex æquatione ultima problematum erui poſſunt regulæ Arithmeticæ, quarum aliquas ad ceterarum exemplum hic ſubjicimus.*

Regula Arithm. 1. Summa quadratorum perpendicularis (a^2) & aggregati hypothenuſæ & baſis (b^2) dividatur per duplum ejus aggregati ($2b$) quotus dabit hypothenuſam *AC*.

Regula Arithm. 2. Ex quadrato aggregati hypothenuſæ & baſis (b^2) ſubtrahe quadratum perpendicularis (a^2), & reſiduum ($b^2 - a^2$) divide per duplum ejuſdem aggregati ($2b$) quotus dabit baſim *B C*.

PROBLEMA II.

Datis in triangulo rectangulo baſi BC *&* *aggregato hypothenuſæ & perpendicularis* AC + AB, *hypothenuſam & perpendicularem invenire.* Fig. 1.

SIt baſis $BC = a$, aggregatum hypothenuſæ & perpendicularis $AC + AB = b$, hypothenuſa vero $AC = x$, erit

erit $AB = b - x$, & *per* 47. *l.* 1. *Eucl.* habetur in terminis analyticis

$$x^2 = b^2 - 2bx + x^2 + a^2$$

$$b^2 - 2bx + a^2$$

$$2bx = a^2 + b^2$$

$$x = \frac{b^2 + a^2}{2b} = AC$$

$$\text{Et } AB (= b - x) = \frac{b^2 - a^2}{2b}$$

Regula Arithm. Summa quadratorum baſis & aggregati hypothenuſæ & perpendicularis ($a^2 + b^2$) diviſa per duplum ejuſdem aggregati ($2b$) dat hypothenuſam *AC*. Differentia vero eorundem quadratorum ($b^2 - a^2$) diviſa per duplum ejuſdem aggregati ($2b$) dat perpendicularem *AB*.

COROLL. *Tam ex primo quam ex ſecundo problemate habetur regula inveniendi in numeris infinita triangula rectangula, quorum latera per numeros integros, & rationales exprimantur. Habentur enim tria latera* $BC = a$, $AC = \frac{a^2 + b^2}{2b}$, & $AB = \frac{b^2 - a^2}{2b}$, *quibus ad eandem denominationem redactis* (*multiplicando* $2 \times 2b$), & *deleto communi denominatore, erunt* $BC = 2ab$, $AC = a^2 + b^2$, & $AB = b^2 - a^2$.

Sumantur itaque ad arbitrium pro a & b duo quicunque numeri. Sit $a = 1$, $b = 2$, erit $2ab = 4$, $a^2 + b^2 = 5$, $b^2 - a^2 = 3$. Latera ergo trianguli rectanguli quæſita ſunt 3, 4, 5.

Sit $a=5$, $b=7$, erit $2ab=70$, $a^2+b^2=25+49=74$, & $b^2-a^2=49-25=24$. Sunt ergo latera quæſita, 24, 70, 74, & ſic de aliis infinitis.

PROBLEMA III.

In triangulo quocunque ABC *datis uno angulo* BAC, *latere* AC *adjacente*, *& reliquorum laterum* AB+BC *ſumma, latus* AB *invenire.* Fig. 2.

EX puncto C ad latus quæſitum AB demittatur perpendicularis CD. In triangulo rectangulo ADC ob datum angulum DAC & hypothenuſam AC, dantur etiam latera AD, & CD *per Probl.* 2. *Trigonom. Tacquet.* Ex data itaque laterum $AB+BC$ ſumma auferatur latus AD jam notum, relinquetur ſumma $BD+BC$ pariter nota. Quæritur latus BD.

Sit laterum $BD+BC$ ſumma data $=c$, latus $CD=$ b, latus quæſitum $BD=x$, erit $BC=c-x$, & $\overline{BD}^2=\overline{BC}^2-\overline{CD}^2$ *per Prop.* 47. *l.* 1. *Eucl.* hoc eſt

$$x^2=c^2-2cx+x^2-b^2$$
$$2cx=c^2-b^2$$
$$x=\frac{c^2-b^2}{2c}=BD$$

Regula Arithm. Inventis per Trigonometriam lateribus AD & CD, ſi quadratum lateris CD (b^2) auferas

ras a quadrato aggregati laterum $BD + BC$ (c^2), & reſiduum ($c^2 - b^2$), dividas per duplum ejuſdem aggregati laterum ($2c$) dabitur latus BD, quod additum lateri jam invento AD, dat latus quæſitum AB.

SCHOL. *Hinc patet, Trigonometriam quoque opem ſuam Algebræ conferre. Nam ſine illa longiori ſane calculo opus fuiſſet. Hoc problemate* Guil. Whiſton (a) *utitur ad calculum Planetarum Geometricum ſupputandum. At prolixiori circuitu hanc ipſam æquationem aſſequitur.*

PROBLEMA IV.

Datis tribus cujuſcunque trianguli lateribus, ſuperficiem invenire. Fig. 3.

PRoducatur, ſi opus ſit, latus BC in D, in quod cadit a vertice A perpendicularis AD. Ponatur autem $AB = a$, $AC = b$, $BC = c$, $AD = x$, $CD = y$, erit $BD = c + y$.

Jam habetur *per Prop.* 47. *l.* 1. *Eucl.*

$\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2$	$\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2$
$x^2 = b^2 - y^2$	$x^2 = a^2 - c^2 - 2cy - y^2$

Proinde $b^2 - y^2 = a^2 - c^2 - 2cy - y^2$

$$2cy = a^2 - b^2 - c^2$$

$$y = \frac{a^2 - b^2 - c^2}{2c}$$

Hinc

(a) Prælect. Aſtronom. Lem. VI. pag. 272. Londini 1707.

Hinc $y^2 = \dfrac{a^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2 + b^4 + c^4}{4c^2}$

Subſtituto valore y^2 in ſuperiori prima æquatione; & extracta deinde radice, oritur

$$AD = \sqrt{\left(b^2 - \frac{a^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2 + b^4 + c^4}{4c^2}\right)}$$

hoc eſt $AD = \sqrt{\left(\dfrac{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4}{4c^2}\right)}$

Quodſi altitudo AD multiplicetur per dimidium baſis BC ($= \frac{1}{2}c$) hoc eſt per cc (ob ſignum radicale *ex Prop.* 4 *Cap.* 4.) erit ſuperficies quæſita

$$\sqrt{\left(\tfrac{1}{8}a^2b^2 + \tfrac{1}{8}a^2c^2 + \tfrac{1}{8}b^2c^2 - \tfrac{1}{16}a^4 - \tfrac{1}{16}b^4 - \tfrac{1}{16}c^4\right)}$$

Regula Aritbmetica ad Geometriam practicam valde utilis, quam auctores communiter tradunt, eſt hujuſmodi: ex laterum ſemiſumma ſubtrahantur ſingula latera, ut habeantur tres differentiæ, quæ inter ſe multiplicentur, earumque productum ducatur in ipſam ſemiſummam laterum: nam hujus producti radix quadrata erit area trianguli quæſita.

Vel ſic: primo multiplicetur ſemiſumma laterum per differentiam unius lateris, & fiat productum A. Secundo multiplicentur inter ſe differentiæ reliquorum duorum laterum, & fiat productum B. Tertio duo illa producta A & B invicem multiplicentur, & extrahatur inde radix quadrata, quæ dabit ſuperficiem trianguli quæſitam.

Semi-

Semiſumma later.	$\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$
Differ. unius later.	$-\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$
Productum A	$-\frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{1}{2}bc + \frac{1}{4}c^2$
Differ. duor. lat.	$-\frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}a$ $+\frac{1}{2}b - \frac{1}{2}c + \frac{1}{2}a$
Productum B	$-\frac{1}{4}b^2 - \frac{1}{4}c^2 + \frac{1}{2}bc + \frac{1}{4}a^2$

Ductis invicem productis $A \times B$, oritur omnino ut ſupra

$$\tfrac{1}{8}a^2b^2 + \tfrac{1}{8}b^2c^2 + \tfrac{1}{8}a^2c^2 - \tfrac{1}{16}a^4 - \tfrac{1}{16}b^4 - \tfrac{1}{16}c^4$$

Cujus radix quadrata dat quæſitam trianguli ſuperficiem.

Sit $a = 15$, $b = 13$, $c = 14$, erit ſumma laterum $= 42$ & ſemiſumma $= 21$. Differentiæ inter hanc & latera ſunt 8, 7, 6, quarum productum $= 336$ ſi multiplicetur per ſemiſummam 21 dat 7056. Cujus radix quadrata 84 dat trianguli ſuperficiem quæſitam.

SHCOL. I. *Verum quo pacto ex ſuperiori æquatione* $\sqrt{(\frac{1}{8}a^2b^2 + \frac{1}{8}a^2c^2 \text{ \&c.})}$ *regula hæc Arithmetica eruatur, non illico apparet. Sed advertatur, tres quantitates* a, b, c, *quæ invicem ſimul multiplicantur, & nuſquam ternæ in producto reperiuntur, neceſſario elidi debere per ſigna negativa & alternatim oppoſita; tertium autem productum oriri ex omnibus illis poſitive ſumptis, ſcilicet*

$$\frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2} \times \frac{a+b+c}{2}$$

Ex harum enim multiplicatione habetur productum ſupra inven-

inventum. Cujus coëfficientes ſracti oriuntur ex eo, quod ſingulæ ſummæ diviſæ ſint per $\frac{1}{2}$, *ut evidens eſt.*

Schol. II. *Porro egregii hujus problematis, teſte* Adriano Metio (*a*), *auctor eſt* Theon Alexandrinus, *magnus Geometra & Aſtronomus.* Dn. de la Hire (*b*) *tanti fecit, ut illud Academiæ regiæ Pariſienſi faciliori ſolutione concinnatum propoſuerit.*

PROBLEMA V.

In dato ſemicirculo ducta perpendiculari DE *ad diametrum* AC, *ducere ab extremitate diametri rectam* AB, *ita ut pars intercepta* FB *æqualis ſit ſemidiametro* AO, *vel* CO. Fig. 4.

Esto factum, ductiſque *BG* parallela perpendiculari *DE*, & recta *BC*, ſit diameter $AC = a$, $AE = b$, $AB = x$, erit $AF = x - \frac{1}{2}a$, cum ſupponatur $FB = AO$, vel $CO = \frac{1}{2}a$.

Ob triangula *BAG* & *EAF* ſimilia erit *AF* . *AE* :: *AB*. *AG*, & in terminis analyticis

$$x - \tfrac{1}{2}a . b :: x . \frac{bx}{x - \frac{1}{2}a} \left(= \frac{2bx}{2x - a}\right) = AG$$

Similiter ob triangula *ABC* & *BAG* ſimilia erit *per Cor. Prop.* 8. *l. Eucl. ex Tacquet.*

CA-

(*a*) Geometr. Pract. part. 2. cap. 3. num. 4. (*b*) Memoires de l'Acad. des ſciences an. 1700. & Acta Erudit. Lipſiæ an. 1704.

$$CA \cdot AB :: AB \cdot AG$$

$$a \cdot x :: x \cdot \frac{2bx}{2x-a}$$

$$x^2 = \frac{2abx}{2x-a}$$

Multipl. per $2x - a$

Divid. per $2x$ $\quad 2x^3 - ax^2 = 2abx$

$$x^2 - \tfrac{1}{2}ax = ab$$

Propoſ. 1. *adde* $\quad \tfrac{1}{16}a^2 \quad \tfrac{1}{16}a^2$

$$x^2 - \tfrac{1}{2}ax + \tfrac{1}{16}a^2 = \tfrac{1}{16}a^2 + ab$$

$$x - \tfrac{1}{4}a = \sqrt{\tfrac{1}{16}a^2 + ab}$$

Extr. rad. $\quad x = \tfrac{1}{4}a + \sqrt{\tfrac{1}{16}a^2 + ab}$

Innoteſcit ergo quantitas chordæ quæſitæ AB, quæ ducenda eſt a puncto A, ut imperatur. Etenim ſi ponatur $a = 4$ & $b = 1$, erit $x\ (= \tfrac{1}{4}a + \sqrt{\tfrac{1}{16}a^2 + ab}) = 1 + \sqrt{5}$, ideoque ipſa chorda $AB = 1 + \sqrt{5}$.

Aliter (Fig. 5.) Completo circulo, producatur DE in F, & ſit diameter $AB = a$, $AE = b$, $DE = c$, $CG = AM$, vel $BM = \tfrac{1}{2}a$, $AC = x$, erit $AG = x - \tfrac{1}{2}a$.

Cum DF ſecta ſit bifariam in E *per Prop.* 3. *l.* 3. *Eucl.*, & non bifariam in G, erit $\overline{DE}^2 = DG \times GF + \overline{GE}^2$ *per Prop.* 5. *l.* 2. ſed $DG \times GF = AG \times GC$ *per Prop.* 35. *l.* 3. & $\overline{GE}^2 = \overline{AG}^2 - \overline{AE}^2$ *per Propoſ.* 47. *l.* 1. ergo $\overline{DE}^2 = AG \times GC + \overline{AG}^2 - \overline{AE}^2$, hoc eſt in terminis analyticis

$$c^2 = x^2 - \tfrac{1}{2}ax - b^2.$$
$$x^2 - \tfrac{1}{2}ax = b^2 + c^2$$

Quæ quidem æquatio reſolvitur *per Prop.* 1. *hujus*, & habetur $x = \tfrac{1}{4}a + \sqrt{\tfrac{1}{16}a^2 + b^2 + c^2}$, quæ a ſuperiori non differt. Nam poſita *AB* (a) = 4, & *AE* (b) = 1, erit *EB* = 3: proinde *DE* (c) = $\sqrt{3}$, cum ſit media proportionalis inter *AE* & *EB per Cor. Prop.* 13. *l.* 6. *Eucl.* Erit ergo $c^2 = 3$, adeoque $\sqrt{\tfrac{1}{16}a^2 + b^2 + c^2} = \sqrt{5}$, & $x = 1 + \sqrt{5}$; atque hinc chorda $AC = 1 + \sqrt{5}$, ut prius.

PROBLEMA VI.

Datis in circuli diametro AB *duobus punctis* D & E *a centro* C *æquidiſtantibus, invenire in peripheria punctum* F, *a quo ductæ* FD, FC, & FE *ſint in continua proportione Geometrica.* Fig. 6.

Esto *F* punctum quæſitum, a quo demittatur perpendicularis *FG*. Sit $DC = CE = a$, $AD = EB = b$, $FC = CA$, vel $CB = r$, $DG = x$, erit $AG = b - x$, & $GB = 2a + b + x$.

$$AG \times GB = 2ab + b^2 - 2ax - x^2 = \overline{FG}^2$$
$$adde\ \overline{GD}^2 = x^2$$

Prop. 47. *l.* 1. $\overline{FD}^2 = 2ab + b^2 - 2ax$

At

At $GE = 2a + x$, & $\overline{GE}^2 = 4a^2 + 4ax + x^2$

adde $\overline{FG}^2 = 2ab + b^2 - 2ax - x^2$

Propoſ. cit. $\overline{FE}^2 = 4a^2 + 2ab + b^2 + 2ax$

Sunt autem ex conditione problematis *FG*, *FC*, & *FD* in continua proportione Geometrica, ergo etiam eorum quadrata *per Theor.* 6. *Cap.* 5., nempe $\overline{FE}^2$, $\overline{FC}^2$, $\overline{FD}^2$, & in terminis analyticis

$$4a^2 + 2ab + b^2 + 2ax . r^2 :: r^2 . 2ba + b^2 - 2ax$$

Fiant porro facilioris calculi gratia quantitates notæ $4a^2 + 2ab + b^2 = m$, & $2ba + b^2 = n$, erit

$$m + 2ax . r^2 :: r^2 . n - 2ax$$

Th. 4 *Cap.* 5. $mn + 2anx - 2amx - 4a^2x^2 = r^4$

Sch. Prop. 1. *Cap.* 5. $4a^2x^2 + 2amx - 2anx = mn - r^4$

Div. per $4a^2$ $x^2 + \frac{m-n}{2a} \times x = \frac{mn - r^4}{4a^2}$

Fiat $\frac{m-n}{2a} = 2p$, & $\frac{mn - r^4}{4a^2} = q$, erit *per Cor. Prop.* 2. *Cap.* 6. æquatio

$$x^2 + 2px - q = 0$$

Quæ quidem *per Prop.* 3. *bujus* facile reſolvitur, eritque nota quantitas *DG*, ſeu punctum *G*, ex quo elevata perpendiculari *GF*, obtinetur in peripheria punctum quæſitum.

Schol. *Problemata Geometrica fere omnia conſtructionem Geometricam admittunt. Nos hoc loco Arithmeticas tantum regulas deduxiſſe contenti Geometricam conſtructionem ad caput ultimum referre ſtatuimus, ne tyrones rerum multitudine & varietate perturbentur.*

CAPUT IX.

De Æquationibus Cubicis.

QUO calculus facilior evadat, ſupponitur jam ſecundus terminus ab æquatione ſublatus: item fractiones, & termini radicales, ut in *Prop.* 2. 8. & 9. *Cap.* 6. docuimus.

Omnes æquationes Cubicæ, ſublato ſecundo termino, ad aliquam ſequentium formularum reducuntur

$$x^3 - px + q = 0$$
$$x^3 - px - q = 0$$
$$x^3 + px - q = 0$$

PROPOSITIO I.

Explicatur æquationum Cubicarum geneſis.

I. SIt æquatio generalis $x^3 - px + q = 0$, repræſentans omnes æquationes cubicas ſecundo termino carentes, quæ, ut ex ſignis apparet, duas habent radices poſitivas, & tertiam negativam duabus illis poſitivis æqualem, alias ſecundus terminus non evaneſceret *per num.* 4. *Prop.* 1. *Cap.* 6.

Sint

Sint igitur duæ radices poſitivæ $f-g$, & $f+g$, erit earum ſumma $=2f$, earumque differentia $=-2g$, & radix negativa $=-2f$. Fiant (mutatis ſignis) æquationes ſimplices $x-f+g=0$, $x-f-g=0$, & $x+2f=0$, ex quarum facto oritur æquatio tertii gradus indeterminata A, tres radices reales continens, duas quidem poſitivas, & tertiam negativam earum ſummæ æqualem, ſcilicet:

$$A \quad \begin{array}{l} x^3 - 3ffx + 2f^3 = 0 \\ \quad\;\; - ggx - 2ggf \end{array}$$

Comparentur hujus termini cum terminis æquationis generalis aſſumptæ $x^3-px+q=0$, cui ex hypotheſi æquatur, fiunt duæ æquationes.

$$1.^{\circ} \quad 3ff+gg=p, \text{ hinc } ff+\tfrac{1}{3}gg=\tfrac{1}{3}p$$
$$2.^{\circ} \quad 2f^3-2ggf=q, \text{ hinc } f^3-ggf=\tfrac{1}{2}q$$

II. Sit æquatio generalis $x^3-px-q=0$, repræſentans omnes æquationes cubicas, in quibus (ut ſigna indicant,) ſunt duæ radices negativæ, & una poſitiva earum ſummæ æqualis, alias ſecundus terminus non evanuiſſet *per num.* 4. *Prop.* 1. *Cap.* 6.

Sint igitur radices negativæ $-f+g$, & $-f-g$: earum ſumma $=-2f$ dat radicem poſitivam illis æqualem, earumque differentia erit $=-2g$. Fiant (mutatis ſignis) æquationes ſimplices $x+f-g=0$, $x+f+g=0$, & $x-2f=0$, ex quarum facto oritur æquatio indeterminata B, continens tres radices reales, duas negativas, & tertiam poſitivam, ſcilicet

$$B \quad \begin{array}{l} x^3 - 3ffx - 2f^3 = 0 \\ \quad\;\; - ggx + 2ggf \end{array}$$

 Cujus

Cujus terminis comparatis cum terminis æquationis generalis aſſumptæ $x^3 - px - q = 0$, cui æqualis ſupponitur, fiunt duæ æquationes.

1.ª $-3ff - gg = -p$, hinc $ff + \frac{1}{3}gg = \frac{1}{3}p$
2.ª $-2f^3 + 2ggf = -q$, hinc $-f^3 + ggf = -\frac{1}{2}q$

Ex utraque æquatione A & B, in quibus f ſupponitur major, quam g (alias $f - g$ non eſſet radix poſitiva, nec $-f + g$ eſſet radix negativa, ut fuit ſuppoſitum) nonnulla inferuntur, ex quorum notitia maxime pendet æquationum cubicarum reſolutio, ſcilicet.

1. Si duæ radices (ſive poſitivæ, ſive negativæ illæ ſint) æquales exiſtant; tunc cubus ex triente quantitatis cognitæ tertii termini eſt æqualis quadrato ex ſemiſſe ultimi termini, hoc eſt $\frac{1}{27}p^3 = \frac{1}{4}qq$.

Nam in hoc caſu in duabus æquationibus ſimplicibus $x - f + g = 0$ & $x - f - g = 0$, earum differentia ſeu quantitas $2g$ fit nulla, hoc eſt $2g = 0$, proinde in æquatione A (idem fit in æquatione B) evaneſcit quantitas $-ggx + 2ggf$, & remanet $x^3 - 3ffx - 2f^3 = 0$. Comparentur hi termini reſidui cum terminis æquationis generalis aſſumptæ $x^3 - px + q = 0$, oriuntur duæ æquationes; prima $-3ff = -p$, ſeu $ff = \frac{1}{3}p$, ſecunda $-2f^3 = +q$, ſeu $f^3 = -\frac{1}{2}q$. Harum prima elevetur ad cubum, & ad quadratum altera: erunt $f^6 = \frac{1}{27}p^3$ & $f^6 = \frac{1}{4}qq$, proinde $\frac{1}{27}p^3 = \frac{1}{4}qq$.

2. Si cubus idem ex triente quantitatis cognitæ tertii termini major fuerit quadrato ex ſemiſſe ultimi termini, hoc eſt ſi $\frac{1}{27}p^3 > \frac{1}{4}qq$, tunc tres radices reales erunt inæquales.

Nam

Nam comparatis terminis æquationis A cum terminis formulæ generalis $x^3 - px + q = 0$, habentur æquationes illæ duæ, prima $ff + \frac{1}{3} gg = \frac{1}{3} p$, ſecunda $f^3 - ggf = \frac{1}{2} q$, & elevando primam ad cubum, oritur

$$f^6 + ggf^4 + \frac{1}{3} g^4 ff + \frac{1}{27} g^6 = \frac{1}{27} p^3$$

Elevando autem ſecundam ad quadratum, habetur

$$f^6 - 2ggf^4 + g^4 ff = \frac{1}{4} qq$$

Si hæc ex illa auferatur, erit reſiduum

$$3ggf^4 - \frac{2}{3} g^4 ff + \frac{1}{27} g^6 = \frac{1}{27} p^3 - \frac{1}{4} qq$$

Jam vero quia ex dictis f major eſt quam g, erit quoque $3ggf^4$ major quam $\frac{2}{3} g^4 ff$ (nam diviſa utraque per $ggff$, quotus $3ff$ major erit quam $\frac{2}{3} gg$) ideoque cum in ſuperiori æquatione $3ggf^4 - \frac{2}{3} g^4 ff + \frac{1}{27} g^6 = \frac{1}{27} p^3 - \frac{1}{4} qq$ unum membrum ſit poſitivum, erit quoque alterum membrum, nempe $\frac{1}{27} p^3 - \frac{1}{4} qq$, ergo $\frac{1}{27} p^3 > \frac{1}{4} qq$.

3. Si idem cubus fuerit minor quadrato ex ſemiſſe ultimi termini, hoc eſt $\frac{1}{27} p^3 < \frac{1}{4} qq$, tunc æquatio duas radices imaginarias continebit. Nam ſi omnes reales eſſent, fieret ex dictis $\frac{1}{27} p^3 = \frac{1}{4} qq$, vel $\frac{1}{27} p^3 > \frac{1}{4} qq$.

Hæc omnia vera exiſtunt, quacunque alia formula generali aſſumpta, ſi pari ratione procedatur.

COROLL. *Ubi non adſunt radices imaginariæ, ſed omnes ſunt reales, tertius terminus ſemper apparet cum ſigno* —, *nempe* — p. *Si duæ radices ſint poſitivæ, & rationales, ultimus terminus erit cum ſigno* + q. *Si negativæ, & rationales, erit* — q. *Contra vero ſi duæ radices irrationales ſint, ultimus terminus erit* ± q.

PRO-

PROPOSITIO II.

In æquationibus cubicis an duæ radices ſint æquales, & quænam ſint, explorare.

I. Sit data æquatio $x^3 - 12x + 16 = 0$. Patet *per Propoſ.* 1. *ejuſque Coroll.* haberi tres radices reales, duas quidem poſitivas & tertiam negativam, quæritur an duæ illæ poſitivæ ſint æquales. Fiat comparatio terminorum cum terminis formulæ generalis $x^3 - px + q = 0$, erit

$$p = 12 \quad \Big| \quad q = 16$$
$$\tfrac{1}{27}p^3 = 64 \quad \Big| \quad \tfrac{1}{4}qq = 64$$

Habentur ergo *per Prop. præc.* duæ radices æquales.

II. Ut una ex illis inveniatur, dividatur triplum ultimi termini per duplum coefficientis tertii termini, quotus dabit radicem quæſitam, hoc eſt $\frac{3q}{2p} = \frac{48}{24} = 2$. Tertia itaque radix erit -4, earum ſummæ æqualis *per num.* 4. *Prop.* 1. *Cap.* 6.

Demonſtr. Nam ex *Prop. præc.* cum radices ſunt æquales, tunc in æquationibus ſimplicibus $x - f + g = 0$ & $x - f - g = 0$ fit $g = 0$; hinc facta ex illis multiplicatione, & comparatis terminis æquationis, quæ inde oritur, cum terminis formulæ generalis, ut in *Prop. cit.* factum eſt, æquatio illa $3ff + gg = p$, fiet $3ff = p$; atque hinc $ff = \frac{1}{3}p$. Similiter $2f^3 - 2ggf = q$, fit $2f^3 = q$, unde oritur $f^3 = \frac{1}{2}q$. Dividatur jam hæc æquatio per illam, hoc eſt primum

mum membrum per primum & ſecundum per ſecundum membrum, erit $f^3 : ff = \frac{1}{2}q : \frac{1}{3}p$, unde $f = \frac{3q}{2p}$, quæ quidem æquatio allatam regulam nobis exhibet.

PROPOSITIO III.

In æquationibus Cubicis an radix aliqua ſit rationalis, & quænam ſit, explorare.

I. Sit æquatio generalis $x^3 - px + q = 0$, quæ tres radices reales habet, duas quidem poſitivas, & tertiam duabus illis æqualem negativam, *per Cor. Prop. 1. hujus.* Quæritur primo, an radix illa negativa, alterutra poſitiva major, ſit rationalis.

Subtrahatur quantitas p ex quadrato proxime majori, & per reſiduum dividatur q. Si quotus fuerit radix ejusdem quadrati aſſumpti, erit ipſe idem radix rationalis quæſita. Si vero ſucceſſive nullum reperiatur quadratum majus p, quod hunc præſtet effectum, ſignum eſt, radicem illam eſſe irrationalem.

Sit exemplum $x^3 - 39x + 70 = 0$ (facta comparatione cum terminis formulæ generalis) erit $p = 39$, $q = 70$. Quadratum proxime majus quam p eſt $= 49$, ex quo ſubtracto p, erit $49 - 39 = 10$: per hoc reſiduum dividatur $q = 70$, quotus 7 æqualis radici quadrati aſſumpti dat radicem rationalem negativam quæſitam.

Explorandum deinde ſit, an aliqua ex radicibus poſitivis ſit rationalis. Sumatur quadratum proxime minus

Ee quam

quam p, quod ſubtrahatur ex ipſo p, & per reſiduum dividatur q. Si quotus fuerit radix ejusdem quadrati aſſumpti, ille dabit radicem rationalem quæſitam. Si vero nullum ſucceſſive quadratum reperiatur minus ipſo p, quod hunc præſtet effectum, ſignum eſt, radices illas eſſe irrationales.

Sit eadem æquatio, quæ prius, $x^3 - 39x + 70 = 0$, cum ſit $p = 39$, $q = 70$, quadratum proxime minus p erit $= 36$, quo ſubtracto ex p, fit $39 - 36 = 3$, & dividendo per hoc reſiduum quantitatem $q = 70$, quotus $= 23\frac{1}{3}$ non eſt radix aſſumpti quadrati.

Sumatur quadratum adhuc minus, nempe 25, quo ſubtracto ex p, erit $39 - 25 = 14$, per quod reſiduum dividendo $q = 70$, habetur quotus $= 5$, æqualis radici quadrati aſſumpti, qui dat unam ex radicibus poſitivis rationalem, nempe 5.

II. Sit æquatio generalis $x^3 - px - q = 0$, quam *ex Propoſ. 1. hujus, ejusque Coroll.* liquet habere tres radices reales, duas quidem negativas & tertiam poſitivam illis æqualem, proinde alterutra negativa majorem. Quæritur, an radix illa poſitiva ſit rationalis. Regula eſt eadem.

Sit exemplum $x^3 - 13x - 12 = 0$, erit $p = 13$, & $q = 12$. Quadratum proxime majus quam p eſt $= 16$, ex quo ſubtrahatur p, erit $16 - 13 = 3$, & dividendo per hoc reſiduum $q = 12$, habetur quotus, qui eſt æqualis lateri quadrati aſſumpti, proinde radix rationalis quæſita $= 4$.

Quæritur deinde, an aliqua ex radicibus negativis ſit rationalis. Sumatur quadratum proxime minus quam p, hoc eſt 9, quod ex ipſo ſubtrahe, erit $13 - 9 = 4$,

per

per quod reſiduum diviſo $q=12$, habetur quotus æqualis lateri quadrati aſſumpti, & radix rationalis quæſita, nempe -3.

Demonſtratio. Poſitis duabus radicibus poſitivis (ut *in Propoſ. 1. hujus*) $f+g$ & $f-g$, earum ſumma $2f$ dat radicem negativam illis æqualem, ejusque quadratum $=4ff$; ex quo ſubtrahatur $3ff+gg$ ($=p$) reſiduum eſt $4ff-3ff-gg$, hoc eſt $ff-gg$; per quod reſiduum ſi dividatur $2f^3-2ggf$ ($=q$) remanet $2f$, radix ſcilicet quadrati $4ff$ & æquationis; unde patet ratio regulæ pro prima parte.

Pro altera parte fiat ex aliqua radice, ſive poſitiva $f-g$, ſive negativa $-f+g$ quadratum $ff-2gf+gg$, quod ſubtrahatur ex $3ff+gg$ ($=p$), reſiduum erit $2ff+2gf$, per quod dividendo $2f^3-2ggf$ ($=q$), quotus dat $f-g$ radicem ſcilicet quadrati & æquationis; unde evidens eſt ratio ſecundæ partis problematis.

Coroll. I. *Tum ex praxi, tum ex demonſtratione patet, ex quadrato aſſumpto ſubtrahi coefficientem* p, *ubi maxima radicum inquiritur: contra vero quadratum aſſumptum ſubtrahi debere ex coefficiente* p, *ubi radix una ex minoribus quæritur. Quod ſi in neutro caſu prodeat ultimi termini* q *diviſor talis, ex cujus diviſione habeatur quotus, qui ſit aſſumpti quadrati latus, ſignum eſt, radices illas quæſitas eſſe irrationales, ut* ex Propoſ. ſequenti *clarius apparebit.*

Coroll. II. *Inventa una ex radicibus rationalibus, ſive illa poſitiva, ſive negativa ſit, dividitur per illam æquatio, quæ ſit ſecundi gradus, & aliæ duæ radices facile innoteſcunt* per Prop. 1. & 3. Cap. 8.

Schol. I. *Si radix inventa, quæ ſit* $= a$, *ponatur duarum reliquarum ſumma, erit illa coefficiens ſecundi termini in æquatione quadratica, quæ reliquas duas radices continet; & ſi per illam dividatur ultimus æquationis datæ terminus* q, *erit* $\frac{q}{a}$ *productum duarum ipſarum radicum in æquatione quadratica contentarum, nempe* $x^2 - ax + \frac{q}{a} = 0$, *ſi radices illæ poſitivæ ſint; vel* $x^2 + ax \frac{q}{a} = 0$, *ſi negativæ; adeoque in primo caſu radix una poſitiva erit* $x = \frac{1}{2}a + \sqrt{(\frac{1}{4}aa - \frac{q}{a})}$, *altera* $x = \frac{1}{2}a - \sqrt{(\frac{1}{4}aa - \frac{q}{a})}$. *In ſecundo caſu una negativa* $x = -\frac{1}{2}a + \sqrt{(\frac{1}{4}aa - \frac{q}{a})}$, *altera* $x = -\frac{1}{2}a - \sqrt{(\frac{1}{4}aa - \frac{q}{a})}$. *Sic in ſuperiori æquatione* $x^3 - 13x - 12 = 0$, *cum inventa fuerit radix poſitiva* $= 4$, *quæ reliquis duabus negativis æqualis eſſe debet, ſi dicatur* a, *erit* $\frac{q}{a} = \frac{12}{4} = 3$, *proinde ex allata formula erit una radix negativa* $x = -2 - \sqrt{4-3} = -3$, *altera* $x = -2 + \sqrt{4-3} = -1$.

Schol. II. *Ubi duplex ſignum occurrit* $\pm$ *vel* $\mp$; *hæc eſt regula: in primo caſu ſumitur ſignum poſitivum, in ſecundo negativum. Plerumque tamen* $\pm$ *ſignificat quantitatem illam tum poſitive, tum negative poſſe uſurpari, ut* in Schol. 2. Prop. 1. Cap. 8. *dictum eſt.*

PRO-

PROPOSITIO IV.

Æquationes cubicas, quæ duas radices imaginarias & realem continent, expedire.

Æquationes cubicas ſecundo termino carentes tunc duas radices imaginarias continere certum eſt:

1. Cum tertius terminus p afficitur quidem ſigno —, ſed $\frac{1}{27}p^3$ minus eſt quam $\frac{1}{4}qq$, *ut in Prop. 1. hujus* dictum eſt, & tunc æquatio generalis erit $x^3 - px \pm q = 0$.

2. Cum tertius terminus ſigno + afficitur, in quo caſu formula generalis eſt $x^3 + px \pm q = 0$.

3. Demum cum æquatio ſecundo & tertio termino caret, (quæ *æquatio pura* tertii gradus dicitur), ut $x^3 \pm q = 0$, tunc autem radix realis ſemper erit $x = \sqrt{\pm q}$.

In ejusmodi autem æquationibus habetur $-q$, ubi radices imaginariæ negativæ ſunt: habetur $+q$, ubi radices imaginariæ ſunt poſitivæ; adeoque in formulis generalibus ponitur utrumque ſignum $\pm q$.

In his reſolvendis quæritur primum radix realis, quæ erit vel poſitiva, vel negativa, ſemper æqualis ſummæ duarum radicum imaginariarum, alias ſecundus terminus tolli non potuiſſet. Ipſa vero radix realis poteſt eſſe rationalis & irrationalis. An ſit rationalis ita detegitur.

I. Sumatur quadratum minus, vel majus quam p, cui addatur $\pm p$; hoc eſt ſi in data æquatione ſit $-p$, ſumatur $-p$, ſi vero ſit $+p$, ſumatur $+p$; & per hoc reſiduum, vel ſummam dividatur q. Si diviſio dat quotum,

 qui

qui ſit radix quadrati aſſumpti, habetur radix rationalis quæſita, ut in primo & ſecundo exemplo ſequenti; alias eſt irrationalis, ut in tertio exemplo.

Data ſit æquatio $x^3 - 1x + 6 = 0$, quæ comparari debet cum formula generali $x^3 - px + q = 0$. Cum autem $\frac{1}{27}p^3 = \frac{1}{27}$ minor ſit quam $\frac{1}{4}qq = 9$, jam manifeſtum eſt, duas radices imaginarias, & quidem poſitivas ob $+q$ in ea deliteſcere. Sumatur quadratum 4, cumque habeatur $-p = -1$, per quadratum $4 - 1 = 3$ dividatur $q = 6$, quotus, qui eſt latus quadrati aſſumpti, dat radicem realem rationalem negativam -2 *per num. 4. Propoſ. 1. Cap. 6.*

Radices imaginariæ habentur *per Prop. 1. & 3. Cap. 8.*, nempe $1 + \sqrt{-2}$, & $1 - \sqrt{-2}$. Nam data æquatio ſi dividatur per radicem rationalem inventam, deprimitur ad ſecundum gradum, ut patet.

II. Data ſit æquatio $x^3 + 27x - 28 = 0$, quæ convenit cum formula generali $x^3 + px - q = 0$, & ſignum $+p$ indicat duas radices imaginarias in ea contineri: quæritur radix rationalis ejusdem. Sumatur quadratum 1, cui addatur $p = 27$, & per hanc ſummam $1 + 27 = 28$ dividatur $q = 28$: quotus 1, cum ſit latus quadrati aſſumpti, dat radicem rationalem quæſitam.

Radices vero imaginariæ (depreſſa æquatione ad ſecundum gradum) habentur *per Propoſ. 1. & 3. Cap. 8.* $-\frac{1}{2} - \sqrt{\frac{1}{4} - 28}$, & $-\frac{1}{2} + \sqrt{\frac{1}{4} - 28}$, negativæ quidem, ut æquationis datæ ſigna indicant, proinde radix realis neceſſario poſitiva eſt.

III. Sit

III. Sit æquatio $x^3 - 6x - 16 = 0$, quæ convenit cum formula generali $x^3 - px - q = 0$, & quia $\frac{1}{27}p^3 = 8$ minus eſt quam $\frac{1}{4}qq = 64$, ſignum eſt, haberi duas radices imaginarias. Videndum, an radix realis ſit rationalis. Sumatur quadratum 4, cui addatur $-p = -6$, & per ſummam $4 - 6 = -2$ diviſo $-q = -16$, habetur $+8$, quod quidem non eſt latus quadrati aſſumpti. Sumatur aliud ſucceſſive quadratum, nempe 9, cui addatur $-p = -6$, & per ſummam hanc $9 - 6 = 3$ dividatur $-q = -16$. Sed hoc cum fieri non poſſit exacte & ſine reſiduo, ſumatur ſucceſſive aliud quadratum majus, nempe 16, quod additum ad $-p = -6$, ſumma fit 10, per quam pariter non eſt diviſibilis ſine reſiduo quantitas $q = 16$. Et cum aliud & aliud quadratum fruſtra ad hunc effectum aſſumatur, ſignum eſt, radicem realem eſſe irrationalem, alia ſane methodo inferius inveniendam *per Prop. 6.*

Demonſtr. fere eadem eſt ac præc. Propoſ. Nam ſi ſupponantur duæ radices imaginariæ negativæ $-f + \sqrt{-3gg}$, & $-f - \sqrt{-3gg}$, ex quibus fiant æquationes ſimplices $x + f - \sqrt{-3gg} = 0$, & $x + f + \sqrt{-3gg} = 0$, earum ſumma $2f$ dabit radicem realem poſitivam, quæ ob ſecundum terminum, quo æquatio caret, illis debet eſſe æqualis. Habetur itaque $x - 2f = 0$. Multiplicentur tres hujuſmodi æquationes, oritur æquatio

$$\begin{aligned} x^3 - 3ffx - 2f^3 &= 0 \\ + 3ggx - 6ggf & \end{aligned}$$

Et facta comparatione cum terminis formulæ generalis $x^3 \pm px - q = 0$, erit

$$-3ff + 3gg = \pm p, \ \& -2f^3 - 6ggf = -q$$

Quod

Quod ſi radices imaginariæ poſitivæ fuerint, ſcilicet $f+\sqrt{-3gg}$, & $f-\sqrt{-3gg}$, tunc radix realis earum ſummæ æqualis ob ſecundum terminum, qui deeſt, neceſſario erit negativa, nempe $-2f$, factaque, ut ſupra, multiplicatione trium æquationum ſimplicium, oritur

$$\begin{array}{l} x^3 - 3ffx + 2f^3 = 0 \\ \quad\;\; + 3ggx + 6ggf \end{array}$$

Comparatis hujus terminis cum terminis formulæ generalis $x^3 \pm px + q = 0$, habetur

$$-3ff + 3gg = \pm p, \; \& \; 2f^3 + 6ggf = q$$

Proinde ſi $4ff$ (quadrato radicis realis $\pm 2f$) addatur $-3ff + 3gg = \pm p$, ſumma erit $4ff - 3ff + 3gg$, hoc eſt $ff + 3gg$, per quam ſi dividatur $-2f^3 - 6ggf = -q$ in primo caſu, vel $2f^3 + 6ggf = q$ in ſecundo caſu, quotus dabit radicem realem quæſitam $\pm 2f$.

Schol. *In æquationibus puris tertii gradus* $x^3 \pm q = 0$ *præter radicem realem* $x = \sqrt[3]{q}$, *vel* $x = \sqrt[3]{-q}$, *dantur quoque aliæ duæ radices, quæ ex dictis in* Prop. *imaginariæ ſunt. Hæ autem ſic inveniuntur. Dividatur*, $x^3 - q = 0$ *per ejus radicem realem inventam* $x - \sqrt[3]{q} = 0$, *quotus* $x^2 + x\sqrt[3]{q} + \sqrt[3]{q^2}$ *erit æquatio ſecundi gradus, quæ duas radices imaginarias continebit. Fiat brevitatis gratia* $q = 8$, *erit* $x^3 - 8 = 0$, *ejuſque radix realis* $x - 2 = 0$, *per quam dividendo* $x^3 - 8 = 0$, *habetur æquatio ſecundi gradus* $x^2 + 2x + 4 = 0$, *cujus radices, ut ex ſignis patet, imaginariæ ſunt, & quidem negativæ,* *nempe*

nempe $-1+\sqrt{-3}$, & $-1-\sqrt{-3}$. *Similiter diviſa æquatione* $x^3+8=0$, *per ejus radicem* $x+2=0$, *invenitur* $x^2-2x+4=0$, *quæ dat duas radices imaginarias poſitivas* $1+\sqrt{-3}$, $1-\sqrt{-3}$. *Sic etiam ſi fiat* $q=1$, *ſeu* $x^3-1=0$, *erunt tres ipſius unitatis radices cubicæ* 1, $-\frac{1}{2}+\sqrt{-\frac{3}{4}}$, & $-\frac{1}{2}-\sqrt{-\frac{3}{4}}$

PROPOSITIO V.

Æquationes Cubicas, in quibus una ſaltem radix eſt rationalis, brevius expedire.

I. Sit æquatio data $x^3+12x=427$, cujus una radix rationalis inquiritur.

Quia incognitæ cubus minor eſt cubo cognitæ, hoc eſt $x^3<427$, nam $x^3=427-12x$; ſumatur radix cubica ipſius 427 proxime minor, nempe 7, ejuſque cubus 343 ſubtrahatur ex ipſa quantitate cognita 427, ſubtracto ex altera parte incognitæ cubo x^3. Ex reſiduo $12x=84$ habetur $x=7$, radix æquationis, aſſumptæ radici cubicæ æqualis, nempe

$$\begin{array}{ll} & x^3+12x=427 \\ \textit{Subtr.} & x^3 \qquad\quad 343 \\ \hline \textit{Divid. per } 12 & +12x=84 \\ & x=7 \end{array}$$

II. Sit æquatio propoſita $x^3-12x=1584$: cum ſit $x^3>1584$, nam $x^3=1584+12x$; ſumatur radix cubica proxime major ipſius 1584, nempe 12, ejuſque cubus

17 8 ſubtrahatur ex ipſo 1584, ſubtracto pariter incognitæ cubo x^3, habetur radix quæſita 12 aſſumptæ radici æqualis

$$\begin{array}{lrl} & x^3 - 12x = & 1584 \\ \textit{Subtr.} & x^3 \quad\quad & 1728 \\ \hline & -12x = & -144 \\ & x = & 12 \end{array}$$

III. Sit æquatio $x^3 + 27x = 28$, ut in præced. Prop. facile cognoſcitur cubum x^3 minorem eſſe cubo ipſius 28: nam $x^3 = 28 - 27x$. Radix cubica proxime minor ejuſdem 28 eſt 3, ejuſque cubus 27, qui ſubductus a 28 relinquit 1: proinde fieret $x = \frac{1}{27}$, qui non eſt valor quæſitus, ut patet. Sumpta radice cubica adhuc minori, nempe 2, factaque cubi ſubtractione, idem ſequitur inconveniens: haberetur enim $x = \frac{20}{27}$. Sumatur itaque pro radice cubica ipſius 28 unitas, ejuſque cubo 1. ſubtracto ex eodem 28, habetur $x = 1$ radix quæſita.

$$\begin{array}{lrl} & x^3 + 27x = & 28 \\ \textit{Subtr.} & x^3 \quad\quad & 1 \\ \hline \textit{Div. per } 27. & +27x = & 27 \\ & x = & \frac{27}{27} = 1 \end{array}$$

Coroll. *Patet ratio, cur in hoc tertio exemplo radix cubica proxime minor ad rem non fuerit; ſed multo minor, imo omnium minima apte fuerit aſſumpta. Nam cubus incognitæ* x^3 *minimam habet rationem ad quantitatem 28. Adduntur enim illi 27x, ut habeatur æquatio. Hoc pro ſimilibus caſibus advertatur.*

Schol-

SCHOL. *Æquationes cubicæ, de quibus hactenus egimus, in quibus ſcilicet una ſaltem radix rationalis inventa fuit, poterant certe reſolvi per ea, quæ diximus in* Cap. 7., *ſed quis non videt hanc viam eſſe breviorem, cum opus non ſit plures ultimi termini diviſores tentare, pluresque diviſiones peragere? ſentiant aliud qui volunt.*

PROPOSITIO VI.

Æquationes Cubicas, quæ duas imaginarias, & realem irrationalem continent, expedire.

I. SIt æquatio $x^3 - px - q = 0$, in qua duas radices imaginarias & realem irrationalem contineri jam conſtet *per Prop.* 4. *&* 5. *hujus*, inveniri debet ipſa radix irrationalis.

1. Subtrahatur $\frac{1}{27}p^3$ ex $\frac{1}{4}qq$, & ex reſiduo extrahatur radix quadrata, cui addatur $\frac{1}{2}q$, dimidium ultimi termini. Et ex hac ſumma extracta radice cubica, habetur prima pars radicis realis irrationalis quæſitæ

$$\sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}$$

2. Dividatur quantitas p per triplum primæ partis radicis jam inventæ, quotus dabit alteram radicis partem priori addendam, ſcilicet

$$\frac{p}{3\sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}}$$

Formula proinde generalis quæſitæ radicis eſt hujuſmodi

$$x=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}+\frac{p}{3\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}$$

Sit exemplum æquatio $x^3-6x-16=0$, in qua per *Prop. præc.* radix realis eſt irrationalis: facta terminorum comparatione cum formula generali, erit $p=6$, & $q=16$, proinde $\tfrac{1}{27}p^3=8$, & $\tfrac{1}{4}qq=64$, hinc $\tfrac{1}{4}qq-\tfrac{1}{27}p^3=64-8=56$.

Hujus reſidui radix quadrata, nempe $\sqrt{56}$, addatur ad $\tfrac{1}{2}q=8$, & ex hac ſumma extrahatur radix cubica, erit prima pars radicis irrationalis quæſitæ $\sqrt[3]{8+\sqrt{56}}$. Et dividendo $p=6$ per triplum hujus primæ partis, obtinetur altera pars radicis, videlicet $\frac{6}{3\sqrt[3]{8+\sqrt{56}}}=\frac{2}{\sqrt[3]{8+\sqrt{56}}}$. Tota itaque radix erit

$$x=\sqrt[3]{8+\sqrt{56}}+\frac{2}{\sqrt[3]{8+\sqrt{56}}}$$

II. Si æquatio data ſit $x^3+px-q=0$ in qua tertius terminus eſt cum ſigno $+$, eadem eſt regula, ſi duo excipias. Primo $\tfrac{1}{27}p^3$ & $\tfrac{1}{4}qq$ addi debent, non ſubtrahi. Secundo radicis inventæ partes non ſunt addendæ, ſed ſecunda ex prima ſubtrahitur, ideoque formula generalis erit

$$x=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}}-\frac{p}{3\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}}}$$

Sit

Sit exemqlum $x^3 + 3x - 26 = 0$, facta comparatione cum ſuperiori formula generali, erit $p = 3$ & $q = 26$, proinde $\frac{1}{27}p^3 = 1$, & $\frac{1}{4}qq = 169$. Hinc $\frac{1}{4}qq + \frac{1}{27}p^3 = 170$, & $\sqrt{\frac{1}{4}qq + \frac{1}{27}p^3} = \sqrt{170}$, quæ radix addatur ad $\frac{1}{2}q = 13$, & ex ſumma extrahatur radix cubica, erit $\sqrt[3]{13 + \sqrt{170}}$ prima pars radicis, & dividendo per hujus triplum quantitatem $p = 3$, habetur ſecunda pars ſubtrahenda ex prima; proinde tota radix erit

$$x = \sqrt[3]{13 + \sqrt{170}} - \frac{1}{\sqrt[3]{13 + \sqrt{170}}}.$$

Demonſtr. Sit ut in *Prop.* 4. æquatio, duas radices imaginarias negativas, & unam poſitivam illis æqualem continens

$$\begin{aligned} x^3 - 3ffx - 2f^3 &= 0 \\ + 3ggx - 6ggf & \end{aligned}$$

factaque comparatione cum terminis formulæ primæ generalis $x^3 - px - q = 0$, erit $-3ff + 3gg = -p$, ſeu $3ff - 3gg = p$, unde $ff - gg = \frac{1}{3}p$, & utriuſque membri cubus

$$A \quad f^6 - 3ggf^4 + 3g^4ff - g^6 = \frac{1}{27}p^3$$

Similiter quia $-2f^3 - 6ggf = -q$, erit $2f^3 + 6ggf = q$, & $f^3 + 3ggf = \frac{1}{2}q$, & quadratum utriuſque membri

$$B \quad f^6 + 6ggf^4 + 9g^4ff = \frac{1}{4}qq$$

Jam ſi ex æquatione *B* ſubtrahatur æquatio *A*, reſiduum erit

$$9ggf^4 + 6g^4ff + g^6 = \frac{1}{4}qq - \frac{1}{27}p^3$$

Extra-

Extractaque radice quadrata, habetur

$$3gff+g^3=\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}$$

Cui addatur ſuperior æquatio $f^3+3ggf=\tfrac{1}{2}q$, fit ſumma

$$f^3+3ggf+3gff+g^3=\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}$$

Ex qua ſi radix cubica extrahatur, prodit prima pars radicis

$$f+g=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}$$

Dividendo autem ſuperiorem æquationem $ff-gg=\tfrac{1}{3}p$ per hanc ultimam, hoc eſt primum membrum per primum, & ſecundum per ſecundum, oritur

$$f-g=\frac{\tfrac{1}{3}p}{\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}$$

Quæ duæ æquationes ſimul junctæ dant realem radicem quæſitam $2f$, nempe

$$2f(=x)=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}+\frac{\tfrac{1}{3}p}{\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}$$

• *Demonſtr.* Quoad ſecundam formulam $x^3+px-q=0$. Sit eadem æquatio, in qua tamen ſupponitur g major f,

$$\begin{aligned}x^3-3ffx-2f^3&=0\\+3ggx-6ggf&\end{aligned}$$

Facta comparatione cum terminis formulæ generalis, erit $3gg-3ff=p$, unde $gg-ff=\tfrac{1}{3}p$, & utriuſque membri cubus

$$M\quad g^6-3ffg^4+3f^4gg-f^6=\tfrac{1}{27}p^3$$

Simi-

Similiter habetur $-2f^3 - 6ggf = -q$, ſeu $f^3 + 3ggf = \frac{1}{2} q$ & utriuſque membri quadratum

$$N \quad f^6 + 6f^4 gg + 9ffg^4 = \tfrac{1}{4} qq$$

Addantur duæ æquationes M & N, erit ſumma

$$g^6 + 6ffg^4 + 9f^4 gg = \tfrac{1}{4} qq + \tfrac{1}{27} p^3$$

Ex qua ſi extrahatur radix quadrata, prodit

$$g^3 + 3ffg = \sqrt{\tfrac{1}{4} qq + \tfrac{1}{27} p^3}$$

Huic adde æquationem ſuperiorem $f^3 + 3ggf = \frac{1}{2} q$, erit

$$g^3 + 3ffg + 2ggf + f^3 = \tfrac{1}{2} q + \sqrt{\tfrac{1}{4} qq + \tfrac{1}{27} p^3}$$

Atque hinc radice tertia extracta, oritur prima pars radicis quæſitæ, nempe

$$g + f = \sqrt[3]{\tfrac{1}{2} q + \sqrt{\tfrac{1}{4} qq + \tfrac{1}{27} p^3}}$$

Ut ſecundam partem obtineas, divide per hanc primam partem æquationem ſuperiorem $gg - ff = \frac{1}{3} p$, ſcilicet

$$\frac{gg - ff}{g + f} = \frac{\frac{1}{3} p}{\sqrt[3]{\frac{1}{2} q + \sqrt{\frac{1}{4} qq + \frac{1}{27} p^3}}}, \text{ erit}$$

$$g - f = \frac{\frac{1}{3} p}{\sqrt[3]{\frac{1}{2} q + \sqrt{\frac{1}{4} qq + \frac{1}{27} p^3}}}$$

Quod ſi hæc ex priori parte radicis ſubducatur, erit radix quæſita

$$2f (= x) = \sqrt[3]{\tfrac{1}{2} q + \sqrt{\tfrac{1}{4} qq + \tfrac{1}{27} p^3}} - \frac{\frac{1}{3} p}{\sqrt[3]{\frac{1}{2} q + \sqrt{\frac{1}{4} qq + \frac{1}{27} p^3}}}$$

Co-

Coroll. *Si in alterutra formula haberetur* + q, *eadem manet demonſtratio, eo ſolum diſcrimine, quod in eo caſu radices imaginariæ eſſent poſitivæ, & radix realis illis æqualis negativa* — 2f

Schol. I. *Per radicem irrationalem inventam dividi debet æquatio, &c. ad ſecundum gradum deprimi, ut radices imaginariæ innoteſcant. Quo id commodius fiat, ſubſtituatur loco ejuſdem radicis irrationalis quantitas rationalis. Sit v. g. æquationis* $x^3 - px - q = 0$ *radix irrationalis inventa* $\sqrt{3}$, *fiat* $\sqrt{3} = m$, *& dividendo per* $x - m$ *ipſam æquationem, oritur æquatio ſecundi gradus* $x^2 + mx + m^2 - p = 0$, *quæ duas radices imaginarias continet, facile inveniendas* per Prop. 1. & 3. Cap. 8. *Reſiduum autem illud* $m^3 - mp - q$, *quod ex hac diviſione habetur, negligitur tanquam zero, ſeu nihil. Hoc recte fieri hinc patet, quod ſi ſubſtituas loco ipſius* m *incognitam* x, *cujus valorem repræſentat, termini æquationis datæ reſtituuntur.*

Schol. II. *Hactenus dicta ſufficerent ad plenam æquationum cubicarum notitiam. Sed æquum non eſt* Cardani *regulas prorſus omittere, quas* Carteſius, Newtonus, Wolfius *& fere omnes recentiores celebrarunt. Earum inventionem* Scipioni Ferreo *Bononienſi* Cardanus *tribuit. Audiunt vulgo* Cardani *regulæ, quod fortaſſe ipſe omnium primus publicaverit. Miror, quod eruditi viri* Fontanellus (*a*), *&* Lamy (*b*) *eaſdem* Varignono *tanquam auctori attribuerint.*

PRO-

(*a*) Hiſtoire de l'Acad. Royale des Scien. 1699. (*b*) Elemens de Mathematique à Paris 1704.

PROPOSITIO VII.

Formulas generales, ſeu regulas Cardani dictas invenire.

I. Sit æquatio generalis $x^3 - px - q = 0$. Fiat $x = f + g$ (fieri debet $f - g$, ubi habetur $+px$) erit

$$x^3 = f^3 + 3ffg + 3ggf + g^3$$

Eſt autem $3ffg + 3ggf = 3fg \times \overline{f+g}$, & ex hypotheſi $x = f + g$, proinde $3fg \times \overline{f+g} = 3fgx$; adeoque $3fgx = 3ffg + 3ggf$, unde

$$x^3 = f^3 + 3fgx + g^3$$
$$x^3 - 3fgx - f^3 - g^3 = 0$$

Comparando jam hujus æquationis terminos cum terminis æquationis generalis aſſumptæ, oriuntur duæ æquationes (duo puncta (:) diviſionem ſignificant) ſcilicet

1ª.	2ª.
$3fg = p$ $g = \frac{1}{3}p : f$ $g^3 = \frac{1}{27}p^3 : f^3$	$f^3 + g^3 = q$

Et ſubſtituendo in ſecunda æquatione valorem ipſius g^3, erit

$$f^3 + \tfrac{1}{27}p^3 : f^3 = q$$

Mult. per f^3

$$f^6 + \tfrac{1}{27}p^3 = qf^3$$

Sch. Prop. 1. Cap. 5.

$$f^6 + qf^3 = -\tfrac{1}{27}p^3$$

Hæc autem æquatio utpote derivativa ſecundi gradus facile reſolvitur *per Prop. 4. Cap. 8.* hoc pacto

$$f^6 - qf^3 = -\tfrac{1}{27}p^3$$
$$\tfrac{1}{4}qq = \tfrac{1}{4}qq$$
$$f^6 - qf^3 + \tfrac{1}{4}qq = \tfrac{1}{4}qq - \tfrac{1}{27}p^3$$

Extr. rad. $f^3 - \tfrac{1}{2}q = \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}$

$$f^3 = \tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}$$

Extr. rad. Cubica $f = \sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}$

Erat autem $f^3 + g^3 = q$, proinde $g^3 = q - f^3$; ſubſtituendo igitur loco f^3 ejus valorem, habetur

$$g^3 = \tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}$$

Extr. rad. Cubica $g = \sqrt[3]{\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}$

Atque hinc $x\ (= f + g) = \sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}} + \sqrt[3]{\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}$.

Quæ quidem eſt formula generalis radicis pro æquatione $x^3 - px - q = 0$.

COROLL. I. *Si æquatio habeat* $+q$, *ut* $x^3 - px + q = 0$, *tunc in formula generali radicis habetur* $-\tfrac{1}{2}q$. *In ceteris nihil differt, nempe* $x = \sqrt[3]{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}} + \sqrt[3]{-\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}p^3}}$.

II. Sit

II. Sit æquatio generalis $x^3+px-q=0$, fiat $x=f-g$ (ob $+px$), erit

$$x^3=f^3-3ffg+3ggf-g^3$$
$$-3ffg+3ggf=-3fg\times\overline{f-g}$$

Sed ex hypotheſi $x=f-g$, ergo $-3fg\times\overline{f-g}=-3fgx$. Proinde ſurrogatis in ſuperiori æquatione $-3fgx$ pro $-3ffg+3ggf$, habetur

$$x^3=f^3-3fgx-g^3$$
$$x^3+3fgx-f^3+g^3=0$$

Comparatis terminis hujus cum terminis æquationis generalis aſſumptæ $x^3+px-q=0$, habentur duæ æquationes

1ª $3fg=p$ $g=\frac{1}{3}p:f$ $g^3=\frac{1}{27}p^3:f^3$	2ª $g^3-f^3=-q$

Ponatur in ſecunda valor ipſius g^3, erit

Mult. per f^3 $\quad \frac{1}{27}p^3:f^3-f^3=-q$

$$\frac{1}{27}p^3-f^6=-qf^3$$

Sch. Prop. 1. Cap. 5. $f^6-qf^3=\frac{1}{27}p^3$

Ex hac æquatione derivativa ſecundi gradus facile extrahitur radix *per Prop. 4. Cap. 8.* nam addendo utrinque $\frac{1}{4}qq$, habetur

$$f^6-qf^3+\frac{1}{4}qq=\frac{1}{4}qq+\frac{1}{27}p^3$$

Extr. rad. $f^3 - \frac{1}{2}q = \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$

$f^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$

Extr. rad. Cubica $f = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$

Erat autem $g^3 - f^3 = -q$, proinde $g^3 = -q + f^3$, & ſurrogato valore ipſius f^3, oritur

$g^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$

Extr. rad. Cubica $g = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$

Atque hinc $x\ (= f - g) = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$ $- \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

Coroll. II. *Quod ſi in æquatione fuerit* $+q$, *ut* $x^3 + px + q = 0$; *tunc in formula generali prima pars radicis habet* $-\frac{1}{2}q$, *eritque* $x = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$ $- \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

PROPOSITIO VIII.

Æquationes cubicas per formulas generales Cardani reſolvere.

Omnes æquationes Cubicæ, ut in principio hujus capitis annotavimus, ad tres ſequentes formulas reduci poſſunt.

I. x^3

$$\begin{array}{ll} \text{I.} & x^3 - px - q = 0 \\ \text{II.} & x^3 - px + q = 0 \\ \text{III.} & x^3 + px - q = 0 \end{array}$$

Singulis autem ſingulæ reſpondent formulæ generales radicis Cubicæ earundem, quas ex Cardani methodo *in præc. Prop.* invenimus, ſcilicet

$$\text{I. } \sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}} + \sqrt[3]{\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}}$$

$$\text{II. } \sqrt[3]{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}} + \sqrt[3]{-\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}}$$

$$\text{III. } \sqrt[3]{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 + \tfrac{1}{27}p^3}} - \sqrt[3]{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 + \tfrac{1}{27}p^3}}$$

Harum formularum uſum celebratiſſimum exempla, quæ ſequuntur, ſatis oſtendent.

I. Sit data æquatio $x^3 - 15x - 4 = 0$. Fiat comparatio cum formula generali $x^3 - px - q = 0$, erit $p = 15$, $\frac{1}{3}p = 5$, & $\frac{1}{27}p^3 = 125$. Item $q = 4$, $\frac{1}{2}q = 2$, & $\frac{1}{4}qq = 4$. Hinc $\sqrt{\frac{1}{4}qq - \frac{1}{27}p^3} = \sqrt{4 - 125} = \sqrt{-121}$, & $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = \sqrt[3]{2 + \sqrt{-121}}$ prima pars radicis. Similiter $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = \sqrt[3]{2 - \sqrt{-121}}$ dat alteram partem. Unde habetur $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$.

Quod ſi ex binomio Cubico $2 + \sqrt{-121}$, & $2 - \sqrt{-121}$ radix extrahatur per methodum inferius tradendam, erit una pars radicis $2 + \sqrt{-1}$, altera $2 - \sqrt{-1}$, ideoque $x = 4$.

Diviſa autem æquatio data per radicem jam inventam, hoc eſt per $x - 4$, deprimitur ad æquationem ſecundi gradus, ejusque radices reliquæ facile eruuntur *per Prop. 1. & 3. Cap. 8.*

II. Sit data æquatio $x^3 - 1x + 6 = 0$. Facta terminorum comparatione, erit $p = 1$, $\frac{1}{3}p = \frac{1}{3}$, & $\frac{1}{27}p^3 = \frac{1}{27}$. Item $q = 6$, $\frac{1}{2}q = 3$, $\frac{1}{4}qq = 9$. Hinc $\sqrt{\frac{1}{4}qq - \frac{1}{27}p^3} = \sqrt{9 - \frac{1}{27}} = \sqrt{\frac{242}{27}}$, & $\sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = \sqrt[3]{-3 - \sqrt{\frac{242}{27}}}$ dat primam radicis partem. At vero $\sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = \sqrt[3]{-3 + \sqrt{\frac{242}{27}}}$ dat alteram; proinde $x = \sqrt[3]{-3 - \sqrt{\frac{242}{27}}} + \sqrt[3]{-3 + \sqrt{\frac{242}{27}}} =$ (radice cubica ex utroque membro extracta *per Prop.10. hujus*) $-1 - \sqrt{\frac{2}{3}}, -1 + \sqrt{\frac{2}{3}} = -2$.

III. Sit data æquatio $x^3 + 3x - 6 = 0$. Fiat comparatio cum formula generali $x^3 + px - q = 0$, erit $p = 3$, $\frac{1}{3}p = 1$, & $\frac{1}{27}p^3 = 1$. Item $q = 6$, $\frac{1}{2}q = 3$, $\frac{1}{4}qq = 9$, proinde $\sqrt{\frac{1}{4}qq + \frac{1}{27}p^3} = \sqrt{9 + 1}$, & $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} = \sqrt[3]{3 + \sqrt{10}}$; atque $x = \sqrt[3]{3 + \sqrt{10}} - \sqrt[3]{-3 + \sqrt{10}}$.

Schol. *Omiſimus cum* Wolfio *quartam formulam cubicam* $x^3 + px + q = 0$, *utpote inutilem, ſeu minus neceſſariam, cujus radix non differt a radice tertiæ formulæ, niſi in hoc tantum, quod in prima parte radicis apponi debet* $-\frac{1}{2}q$, *in ſecunda vero* $+\frac{1}{2}q$. *Imo ad ſolvendam quan-*

quancunque æquationem cubicam, cui deſit ſecundus terminus, ſufficit prima Cardani *formula, qua una tanquam generali Canone poteſt quis ſatis commode uti.*

PROPOSITIO IX.

Idem problema brevius abſolvitur.

Sit æquatio generalis cubicas omnes repræſentans, quæ ſecundo termino carent, $x^3 + px + q = 0$. Fiat $x = a + b$ (a & b ſunt indeterminatæ) erit

$$x^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$3a^2b + 3ab^2 = 3ab \times \overline{a+b}, \ \& \ x = a + b$$

Proinde $\quad 3a^2b + 3ab^2 = 3abx$

Adeoque $\quad x^3 = a^3 + 3abx + b^3$

Poſito hoc valore loco ipſius x^3 in æquatione generali, habetur

$$a^3 + 3abx + b^3 + px + q = 0$$

Fiant duæ æquationes, quarum altera contineat quantitates radicales, altera rationales, & utraque æquetur zero.

1ª. $3abx + px = 0$ 2ª. $a^3 + b^3 + q = 0$

Ex prima habetur $b = -\dfrac{p}{3a}$, & $b^3 = -\dfrac{p^3}{27a^3}$. Hoc valore poſito in ſecunda æquatione, oritur

Mult.

Mult. per a^3 $\quad a^3 - \dfrac{p^3}{27a^3} + q = 0$

$$a^6 + qa^3 = \frac{p^3}{27}$$

Et reſoluta hac æquatione derivativa ſecundi gradus, addendo utrinque $\frac{1}{4}qq$, *per Prop. 4. Cap. 8.* habetur

$$a^3 = -\tfrac{1}{2}q \pm \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}p^3}$$
$$a = \sqrt[3]{-\tfrac{1}{2}q \pm \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}p^3}}$$

Erat autem $b = -\dfrac{p}{3a}$, proinde ſecunda pars radicis (ob $x = a + b$) obtinebitur dividendo $-p$ per triplum primæ partis radicis inventæ, hoc eſt per $3a$, eritque $x = \sqrt[3]{-\frac{1}{2}q \pm \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} - \dfrac{p}{3\sqrt[3]{-\frac{1}{2}q \pm \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}}$.

I. Sit æquatio $x^3 - 7x - 6 = 0$, erit $p = -7$, & $\dfrac{p^3}{27} = -\frac{343}{27}$. At $q = -6$, & $\frac{1}{4}qq = 9$, hinc invenitur $x = \sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}} + \dfrac{7}{3\sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}}}$.

II. Sit æquatio $x^3 - 6x + 8 = 0$: comparatis terminis cum terminis formulæ generalis, erit $p = -6$ & $\frac{1}{27}p^3 = -8$: item $q = 8$ & $\frac{1}{4}qq = 16$. Hinc habetur $x = \sqrt[3]{-4 \pm \sqrt{8}} + \dfrac{2}{\sqrt[3]{-4 \pm \sqrt{8}}}$.

COROLL.

COROLL. *Hæ formulæ radicis generales conveniunt, ut patet, cum formulis, quas* in Prop. 6. hujus *alia via tradidimus. Sunt hæ quidem multo ſimpliciores illis, quas in ſuperiori* Prop. 7. *explicavimus, cum hæ unicam radicis extractionem, illæ duas requirant.*

SCHOL. I. *At dubitant hic tyrones, an per diverſas ejuſmodi formulas eadem omnino radix obtineatur. An ex. gr. ſuperioris primæ æquationis* $x^3 - 7x - 6 = 0$ *radix per formulam Propoſ. hujus expreſſa* $\sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}} + \frac{7}{3\sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}}}$ *ſit omnino eadem cum æquationis ejuſdem radice, quæ per primam* Cardani *formulam haberetur, nempe* $\sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$, *hoc eſt an ſecunda hujus pars ſecundæ parti alterius formulæ ſit æqualis. Sed nullum eſt prorſus dubium. Ponatur enim* $\frac{7}{3\sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}}}$, *ſeu* $\frac{\frac{7}{3}}{\sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}}} = \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$ *erit ex communi lege æquationum* $\frac{7}{3} = \sqrt[3]{3 - \sqrt{-\frac{100}{27}}} \times \sqrt[3]{3 \pm \sqrt{-\frac{100}{27}}}$, *factaque radicalium multiplicatione* per Prop. 12. Cap. 4. *invenitur* $\frac{7}{3} = \sqrt[3]{\frac{343}{27}} = \frac{7}{3}$. *Et quidem extracta radice tertia ex binomio cubico* $3 \pm \sqrt{-\frac{100}{27}}$ per Prop. 10. hujus, *habetur* $1 \pm \sqrt{-\frac{4}{3}}$, *adeoque* $1 + \sqrt{-\frac{4}{3}} \times 1 - \sqrt{-\frac{4}{3}} = 1 + \frac{4}{3} = \frac{7}{3}$, *ut prius; proinde evidens eſt, ex utraque formula eandem radicem obtineri.*

SCHOL. II. *Si in æquatione cubica radix aliqua rationalis*

nalis exiſtat, tunc ejuſmodi formulas inutiles quidem eſſe cenſeo, cum talis æquatio via breviſſima reſolvi poſſit per Prop. 4. vel 5. hujus; *& abſurdi genus ſit, radicem rationalem per terminos radicales, imo etiam per radices imaginarias inquirere. Quod ſi æquatio duas radices imaginarias continet & unam realem irrationalem, tunc hujuſmodi formulæ opportune adhibentur.*

Schol. III. *Ceterum ubi in æquatione* $x^3 - px \pm q = 0$ *habetur* $\frac{1}{27}p^3 > \frac{1}{4}qq$, *adeoque tres radices reales, & inæquales ſunt* per Prop. 1. num. 2. *& nulla earum rationalis invenitur* per Propoſ. 3. vel 5., *tunc neceſſario* $\sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$ *erit quantitas imaginaria. Proinde radices, quæ ex natura æquationis reales eſſe debent, tanquam imaginariæ per* Cardani, *aliorumque formulas nobis exhibentur. Quod plane eſt abſurdum. Et licet ſagaciſſimum Analyſtarum ingenium, ut huic malo occurreret, ex binomio Cubico radices imaginarias involvente radicem realem erui poſſe docuerit, quod & nos in ſequenti Propoſ. explicabimus, id tamen nodum non ſolvit. Nam hoc tunc ſolum bene eſt, cum æquatio unam ſaltem radicem realem & ſimul rationalem continet: ſed ubi nulla rationalis eſt, & omnes reales, tunc ars deficit, nullaque ratione per Analyſim radices ejuſmodi exprimi poſſunt. Et hic dicitur* caſus irreductibilis, & irreſolutus; *in quo opus eſt ad Geometriam confugere, quæ radices illas per lineas exprimit: quemadmodum ipſa rationem, quam quadrati latus ad diametrum habet incommenſurabilem, quæque numeris explicari non poteſt, per lineas opportune deſignat.*

PRO-

PROPOSITIO X.

Ex binomio Cubico radicem extrahere.

EXtrahenda ſit radix cubica ex binomio $20 + \sqrt{392}$. Reducatur pars radicalis $\sqrt{392}$ ad ſimpliciorem expreſſionem *per Cor.* 1. *Prop.* 2. *Cap.* 4. fiet $14\sqrt{2}$.

Certum eſt, binomiæ radicis partem unam fore $\sqrt{2}$, vel hujus multiplum (ut in ſecundo exemplo) per numerum rationalem exprimendum, quem voco m. Nam ſi $\sqrt{2}$ non ingrederetur radicem, neque cubum ingrederetur, ut manifeſtum eſt.

Ponatur itaque binomii cubici $20 + 14\sqrt{2}$ pars rationalis $20 = a$, irrationalis autem $14\sqrt{2} = m\sqrt{c}$, erit tota radix $a + m\sqrt{c}$, ex qua (a, c, m ſunt indeterminatæ) fiat cubus

$$a^3 + 3a^2 m\sqrt{c} + 3\,mmac + m^3 c\sqrt{c}$$

Cujus pars rationalis eſt $a^3 + 3\,mmac$; pars irrationalis $3a^2 m\sqrt{c} + m^3 c\sqrt{c}$. Quia vero $\sqrt{c} = \sqrt{2}$, erit

$$3a^2 m\sqrt{c} + m^3 c\sqrt{c} = 3a^2 m\sqrt{2} + m^3\, 2\sqrt{2} = 14\sqrt{2}.$$

Poſito $m = 1$, & dividendo per $\sqrt{2}$, habetur

$$3a^2 + 2 = 14$$
$$3a^2 = 12$$
$$a^2 = 4$$
$$a = 2$$

Valore autem ipſius a in altera radicis parte ſubſtituto, erit

$$a^3 + 3mmac = 8 + 12 = 20$$

 Quod

Quod quidem cum hypotheſi optime convenit, proinde radix cubica $a+m\sqrt{c}=2+\sqrt{2}$.

II. Sit extrahenda radix cubica ex binomio $\sqrt{18252}-135$. Reducatur *per Prop.* 2. *Cap.* 4. ad ſimplicem expreſſionem, fiet $78\sqrt{3}-135$.

Ponatur radix eſſe $m\sqrt{c}-a$ (hoc eſt $78\sqrt{3}=m\sqrt{c}$ & $-135=-a$) & fiat cubus

$$m^3c\sqrt{c}-3m^2ac+3a^2m\sqrt{c}-a^3$$

Cum ſit $\sqrt{3}=\sqrt{c}$, pars irrationalis erit

$$m^3c\sqrt{c}+3a^2m\sqrt{c}=3m^3\sqrt{3}+3a^2m\sqrt{3}=78\sqrt{3}.$$

Poſito autem $m=1$, & dividendo per $\sqrt{3}$, erit

$$3+3a^2=78$$
$$3a^2=75$$
$$a^2=25$$
$$a=5$$

Quo valore ſurrogato in altera radicis parte, habetur

$$-a^3-3m^2ac=-125-45=-170$$

Quod cum hypotheſi minime convenit, deberet enim eſſe -135; proinde ſignum eſt m ſumptum eſſe juſto minorem.

Ponatur $m=2$, adeoque $m\sqrt{c}=2\sqrt{3}$, erit

$$m^3c\sqrt{c}+3a^2m\sqrt{c}=24\sqrt{3}+6a^2\sqrt{3}=78\sqrt{3}$$

Div. per $\sqrt{3}$) $24+6a^2=78$

$$6a^2=54$$
$$a^2=9,\text{ hinc } a=3$$

Proin-

Proinde pars rationalis $-a^3-3m^2ac=-27-108$ $=-135$. Quod quia cum hypotheſi recte convenit, infertur, radicem cubicam $m\sqrt{c}-a=2\sqrt{3}-3$.

III. Sit binomium $3+\sqrt{-\frac{100}{27}}$, cujus radix cubica quæritur. Habetur *per Propoſ. 2. Cap. 4.* $\sqrt{-\frac{100}{27}}=\frac{5}{3}\sqrt{-\frac{4}{3}}$, Hinc ponatur $3+\frac{5}{3}\sqrt{-\frac{4}{3}}=a+m\sqrt{c}$ radici, ex qua fiat Cubus

$$a^3+3a^2m\sqrt{c}+3am^2c+m^3c\sqrt{c}$$

Et quia $\sqrt{c}=\sqrt{-\frac{4}{3}}$, pars radicalis erit

$$3a^2m\sqrt{c}+m^3c\sqrt{c}=3a^2m\sqrt{-\frac{4}{3}}+m^3\times-\frac{4}{3}\sqrt{-\frac{4}{3}}.$$

Poſito $m=1$, & dividendo per $\sqrt{-\frac{4}{3}}$, oritur

$$3a^2-\frac{4}{3}=\frac{5}{3},\ \&\ 3a^2=\frac{9}{3}$$
$$a^2=\frac{9}{9},\ \&\ a=\pm 1$$

Quo valore ſubſtituto, habetur pars rationalis

$$a^3+3m^2ac=-1+4=3$$

Quod convenit cum hypotheſi, unde radix quæſita $a+m\sqrt{c}=1+\sqrt{-\frac{4}{3}}$.

IV. Extrahenda ſit radix cubica ex binomio, cujus utraque pars eſt irrationalis $\sqrt{243}+\sqrt{242}$, quod *per Propoſ. 2. Cap. 4.* fit $9\sqrt{3}+11\sqrt{2}$.

Ponatur prima pars radicis $9\sqrt{3}=m\sqrt{c}$, ſecunda vero $11\sqrt{2}=n\sqrt{d}$, adeoque tota radix erit $m\sqrt{c}+n\sqrt{d}$, & cubus ex illa factus

$$m^3c\sqrt{c}+3m^2nc\sqrt{d}+3mn^2d\sqrt{c}+n^3d\sqrt{d}$$

Atque hinc ob $m\sqrt{c}=9\sqrt{3}$, erit una pars

$$m^3 c\sqrt{c}+3mn^2 d\sqrt{c}=3m^3\sqrt{3}+6mn^2\sqrt{3}=9\sqrt{3}$$

Poſito autem $m=1$, & dividendo per $\sqrt{3}$, habetur

$$3+6n^2=9, \ \& \ 6n^2=6$$
$$\text{hinc } n=1$$

Ideoque hoc valore in altero membro ſubſtituto, erit

$$3m^2nc\sqrt{d}+n^3d\sqrt{d}=9\sqrt{2}+2\sqrt{2}=11\sqrt{2}$$

Quod ſane cum hypotheſi convenit, unde radix cubica quæſita $m\sqrt{c}+n\sqrt{d}=\sqrt{3}+\sqrt{2}$.

Corol. *Cum ſubſtituto valore provenit numerus numero rationali major (ut in ſecundo exemplo pro* — 135 *proveniebat* — 170) *ſignum eſt, valorem ipſum* m *ſumptum eſſe juſto minorem, adeoque ſumendum alium illo majorem. Sin autem pro* — 135 *proveniſſet numerus minor, tunc eſſet indicium* m *ſumptum eſſe juſto majorem; proinde ſi in primo caſu fuit* m $=1$, *& in ſecundo* m $=2$, *ſumi debet intermedium, hoc eſt* m $=1\frac{1}{2}$. *Quo facto, ſi res nec bene ſuccedat, ſignum eſt, binomii radicem, qualis quæritur, non haberi. Ceterum valor ipſius* m *ſemper debet eſſe vel numerus integer, vel ſaltem integri dimidius.*

Schol. I. *Si* m *non eſt pars aliquota quantitatis, quæ ſigno radicali præfigitur, radix cubica non obtinetur quod advertatur.*

Schol. II. *Præclara hæc methodus eſt* Jo. Walliſii (a), *quæ, ut ipſe ait, eſt ceteris longe ſimplicior. Nam quæ primum*

(a) Algebra Cap. 47. Oper. Tom. 2.

mum a Franciſco Schootenio (*a*), *deinde* a Jacobo Ozanam (*b*) *aliiſque inde ſcriptoribus tradita eſt regula, prolixior eſt, & plures exigit cautiones. Quæ autem traditur a* Wolfio (*c*) *per analyſim, non modo prolixior eſt, ſed æquationis cubicæ ſolutionem involvit, cujus radix non ita facile occurrit; imo illa, niſi ſit rationalis, inveniri non poteſt. Ex hac vero methodo cito certi ſumus, an radix quæſita ex binomio dato extrahi poſſit.*

SCHOL. III. *Pro facili & expedito examine radicis ex binomio cubico inventæ, ſatis erit inveſtigare ejus partem rationalem tantum: quod ſit addendo cubum partis rationalis ad triplum ejuſdem partis multiplicatæ per quadratum partis alterius. Sic ex binomio* $78\sqrt{3} - 135$ *ſit radix inventa* $2\sqrt{3} - 3$. *Adde cubum partis rationalis* -3, *ſcilicet* -27, *ad triplum ejuſdem partis* -9 *multiplicatæ per quadratum partis alterius* 12, *nempe ad* -108, *ſumma dat* -135: *quod quia convenit cum parte rationali binomii dati, conſtat* $2\sqrt{3} - 3$ *eſſe veram radicem. Ceterum radix cubica per ſuperiorem* Wallisii *methodum inventa hoc examine probari non indiget, cum de ipſa aliunde certi ſimus.*

SCHOL. IV. *Quod ſi e binomio cubico radix extrahi nullo modo poſſit, & æquationis datæ radices per* Cardani *regulas expreſſæ valde implicatæ appareant, tunc præſtat, ut* Franciſcus Schootenius (*d*) *opportune admonet, æquationem illam ex ſola ſua conſtitutione innuere, quam ipſam confuſe per eaſdem* Cardani *formulas exprimere. Sic data æqua-*

(*a*) Comment. Geometr. Carteſ. edit. 3. pag. m. 389. (*b*) Nouveaux Elemens d' Algeb. Cap. 5. Probl. VIII. (*c*) Elementa Analyt. edit. 3. Probl. 169. (*d*) Comment. in lib. 3. Geom. Carteſ.

æquatione $x^3 = 8x + 24$, *ejus naturam facilius concipies, ſi dicas, radicem hujus æquationis* x *talem eſſe, ut ſi multiplicetur per* 8, *&* *inſuper addantur* 24, *æqualis ſit futura ſuo cubo* x^3, *quam ſi eandem per* Cardani *formulam primam hoc modo exprimas* $x = \sqrt[3]{12 + \sqrt{125\frac{1}{27}}} + \sqrt[3]{12 - \sqrt{125\frac{1}{27}}}$. *Et ſic de aliis.*

PROBLEMATA CUBICA.

PROBL. I.

Data ſumma duorum cuborum & differentia laterum, latera ipſa invenire.

Sit ſumma cuborum $= 2a$, latera $= 2x$, quorum differentia ſit $= 2b$, erit latus majus $x + b$, minus $x - b$ *per Theor.* 3 *Cap.* 5. Cubi vero ex ipſis dant æquationem, quæ ſequitur

$$2x^3 + 6b^2 x = 2a$$

Div. per 2.
$$x^3 + 3b^2 x = a$$
$$x^3 + 3b^2 x - a = 0$$

Si fiat $a = 14$, $b = 1$, erit æquatio determinata

$$x^3 + 3x - 14 = 0$$

Quæ reſolvitur *per Propoſ.* 4. *hujus.* Nam aſſumendo quadratum 4, erit $4 + 3 = 7$, per quem dividendo 14, quotus 2, (radix quadrati aſſumpti) dat radicem æquationis quæſitam. Erit ergo unum latus $x + b = 2 + 1$, alterum $x - b = 2 - 1$.

Bre-

Brevius reſolvitur *per Prop.* 5. *hujus.* Nam ſumpta radice cubica proxime minori ipſius 14, nempe 2, fit cubus 8, & facta hinc inde utriuſque cubi ſubtractione, habetur ſtatim $x=2$

Subtr.

$$\begin{array}{r} x^3+3x=14 \\ x^3 \qquad 8 \\ \hline +3x=6 \\ x=2 \end{array}$$

PROBLEMA II.

Data ſumma duorum cuborum & rectangulo ſub lateribus, invenire ipſa cuborum latera.

SIt Cuborum ſumma $=2a$, rectangulum ex lateribus $=2b$, latus unum quæſitum $=x$, & alterum $=y$, erit $x^3+y^3=2a$, & $xy=2b$; hinc $y=\frac{2b}{x}$, & $y^3=\frac{8b^3}{x^3}$ Et ſubſtituto in prima æquatione loco y^3 ejus valore, habetur

Mult. per x^3

$$\begin{array}{c} x^3+\frac{8b^3}{x^3}=2a \\ \hline x^6+8b^3=2ax^3 \\ x^6-2ax^3=-8b^3 \end{array}$$

Quæ cum ſit derivativa ſecundi gradus, ſi fiat $x^3=z$, erit *per Prop.* 4. *Cap.* 8. æquatio

$$z^2-2az=-8b^3$$

Si ſupponatur $3a = 72$, & $2b=8$, æquatio convertitur in ſequentem

$$z^2 - 72z = -512$$

Prop. 1. *Cap.* 8. *adde* $1296 \quad 1296$

Extr. rad. $z^2 - 72z + 1296 = 784$

$$z - 36 = \sqrt{784}\ (=28)$$

$$z = 36 + 28 = 64$$

Sed poſita fuit $x^3 = z$, ergo $x = \sqrt[3]{z} = \sqrt[3]{64} = 4$. Eſt ergo $x = 4$, & $y\ (= \frac{2b}{x}) = \frac{8}{4} = 2$.

PROBLEMA III.

Invenire tres numeros Arithmetice proportionales, quorum differentia & ſolidum ab illis factum ſunt data.

Sit trium proportionalium quæſitorum differentia data $= d$, ſolidum $= s$, & numerorum quæſitorum primus $= x$, erit ſecundus $= x+d$, tertius $= x + 2d$ *per Schol.* 1. *Prop.* 3. *Cap.* 5., & ſolidum ab illis factum

$$x^3 + 3dx^2 + 2ddx = s$$

Auferatur ſecundus terminus *per Propoſ.* 5. *Cap.* 6. facto $x = y - d$, oritur æquatio

$$y^3 * - ddy = s$$

Proinde ſi ſupponatur $d = 3$, & $s = 28$, æquatio eadem erit

y^3

$$y^3 * - 9y = 28$$
$$y^3 - 9y - 28 = 0$$

Hæc autem facile reſolvitur *per Prop.* 6. *hujus.* Fiat enim $p = 9$, $q = 28$; ex formula generali $x^3 - px \pm q = 0$ invenitur $y = \sqrt[3]{14 + \sqrt{169}} + \frac{3}{\sqrt[3]{14 + \sqrt{169}}}$. Ex hoc binomio cubico extracta radice *per Prop.* 10. *hujus*, habetur $y = 2 + \sqrt{1} + \frac{3}{2 + \sqrt{1}}$, factaque reductione, hoc eſt multiplicando primam partem per denominatorem alterius, & addendo illi $\frac{1}{3}p$, ſeu (in hoc exemplo) 3, habetur $\frac{8 + 4\sqrt{1}}{2 + \sqrt{1}}$. Fiat actu diviſio *per Prop.* 6. *Cap.* 4. erit quotus 4. Eſt ergo $y = 4$; ſed poſita fuit $x = y - d$, unde erit $x = 4 - 3 = 1$. Sunt ergo numeri proportionales quæſiti 1, 4, 7, quorum ſolidum $= 28$.

COROLL. *Poterat quoque Problema reſolvi* per Prop. 4. vel 5. hujus, *cum unam radicem rationalem habeat, ut patet: imo &* per Propoſ. 1. Cap. 7. *cum habeatur* 4 *unus ex ultimi termini diviſoribus, poſitoque* $y = 4$, *tota æquatio evaneſcat. Malui tamen* per Prop. 6. hujus *illud reſolvere, ut tyrones diſcant radices ex binomio Cubico extractas reducere, & alia, in quibus difficultatem quandoque ſolent experiri.*

PROBLEMA IV.

Numerum 10 ita dividere in quatuor partes Geometrice continue proportionales, ut ſi prima ducatur in 8, ſecunda in 4, tertia in 3, & quarta in 1. ſumma omnium fiat 16.

SIt una dati numeri pars $= x$ & proportionis Geometricæ denominator $= y$, erit ex prima problematis conditione $x + xy + xy^2 + xy^3 = 10$, & ex ſecunda $8x + 4xy + 3xy^2 + xy^3 = 16$. Sumptoque ex utraque æquatione valore ipſius x, erit

$$x = \frac{10}{1 + y + y^2 + y^3}, \quad x = \frac{16}{8 + 4y + 3y^2 + y^3}$$

Proinde $$\frac{10}{1 + y + y^2 + y^3} = \frac{16}{8 + 4y + 3y^2 + y^3}$$

Unde $$y^3 - \tfrac{7}{3} y^2 - 4y - \tfrac{32}{3} = 0$$

$$\begin{array}{cccc} 1. & 3. & 9. & 27 \\ \hline \end{array}$$

$$y^3 - 7y^2 - 36y - 288 = 0$$

Et dividendo per $y - 12$, nihil remanet, adeoque $y = 12$. Sed cum radix ſuperioris æquationis ad fractiones tollendas multiplicata ſit per progreſſionem Geometricam triplam, erit 12 triplum quæſitæ radicis; hinc poſito in alterutra æquatione $y = 4$, erit $x = \frac{2}{17}$ prima pars quæſita, reliquæ vero $\frac{8}{17}$, $\frac{32}{17}$, $\frac{128}{17}$, quarum ſumma $\frac{170}{17} = 10$,

& ſi

& ſi ſingulæ ducantur in numeros imperatos, erit earum ſumma $\frac{272}{17} = 16$, ut patet.

SCHOL. *Problematis hujus auctor eſt* Lucas Paciolus (a) *a quo fuit particulari ratione per numeros determinatos ſolutum. Generalem ejuſdem ſolutionem per Algebram tentavit* Joannes Camillus Glorioſus (b), *ſed cum immenſo pene calculo plures paginas implevifſet, opus omnino non abſolvit. Concinna hæc ſolutio* ab Antonio de Monforte (c) *tradita eſt.*

PROBLEMA V.

Datis duabus rectis, alteram earum ita producere, ut alterius quadratum ad quadratum partis productæ eandem rationem habeat, quam pars producta ad totam rectam.

SInt rectæ datæ a & b, & producatur b, cujus pars producta ſit x, tota recta erit $b + x$, & ex conditione probl.

$$a^2 \cdot x^2 :: x \cdot b + x$$

Theor. 4. *Cap.* 5. $x^3 = a^2 b + a^2 x$

$$x^3 - a^2 x - a^2 b = 0$$

Sit $a = 2$, $b = 12$, æquatio fiet

$$x^3 - 4x - 48 = 0$$

Cujus reſolutio facile obtinetur *per Prop.* 4. *vel* 5. *hujus*, ex

(a) Arithm. Quæſt. 27. (b) Exercitat. Mathemat. 4. Neap. 1635. (c) Tract. de Probl. determinat. Neap. 1690.

ex quibus innoteſcit $x=4$, unde $b+x=16$. Quod ſi ponatur $a=2$, $b=5$, fiet æquatio, quæ nonniſi per Cardani regulas expeditur, nempe

$$x^3 - 4x - 20 = 0$$

Erit igitur $p=4$, $\frac{1}{3}p=\frac{4}{3}$, & $\frac{1}{27}p^3=\frac{64}{27}$. Item $q=20$, $\frac{1}{2}q=10$, $\frac{1}{4}qq=100$. Hinc invenitur

$$x = \sqrt[3]{10+\sqrt{\tfrac{2636}{27}}} + \sqrt[3]{10-\sqrt{\tfrac{2636}{27}}}$$

Hujus autem binomii cubici radix (ſiquidem extrahi poſſit) obtinebitur *per Prop.* 10 *hujus*.

PROBLEMA VI.

Invenire triangulum ABC, *cujus latera* AB, AC, BC *&* *perpendiculum* DC *ſint in Arithmetica proportione.* Fig. 7.

Ponatur perpendiculum $CD=x$, differentia laterum trianguli $=d$, erit $BC=x+d$, $AC=x+2d$, $AB=x+3d$ *per Schol.* 1. *Propoſ.* 3. *Cap.* 5. Eſt autem $\overline{AD}^2=\overline{AC}^2-\overline{CD}^2$, & $\overline{BD}^2=\overline{BC}^2-\overline{CD}^2$ *per Prop.* 47. *l.* 1. *Eucl.* erunt proinde $AD=\sqrt{(\overline{AC}^2-\overline{CD}^2)}$, & $BD=\sqrt{(\overline{BC}^2-\overline{CD}^2)}$; adeoque $AD=\sqrt{4dx+4d^2}$, & $BD=\sqrt{2dx+d^2}$, proinde tota AB $(=x+3d)=\sqrt{4dx+4d^2}+\sqrt{2dx+d^2}$.

Fiat

Fiat calculi commodo $x + 3d = m$, erit

$$m = \sqrt{4dx + 4d^2} + \sqrt{2dx + d^2}$$

$$m - \sqrt{4dx + 4d^2} = \sqrt{2dx + d^2}$$

Elevetur utrumque membrum ad quadratum, ut radicales exterminentur *per Reg.* 4. *Prop.* 1. *Cap.* 5. erit

$$m^2 - 2m\sqrt{4dx + 4d} + 4dx + 4d^2 = 2dx + d^2$$

$$m^2 - 2m\sqrt{4dx + 4d^2} + 2dx + 3d^2 = 0$$

$$m^2 + 2dx + 3d^2 = 2m\sqrt{4dx + 4d^2}$$

Elevato rurſus ad quadratum utroque hujus æquationis membro, & facta terminorum reductione *per Lem. Cap.* 1. habetur

$$m^4 - 12dxm^2 - 10d^2m^2 + 4d^2x^2 + 12d^3x + 9d^4 = 0$$

Subſtitutis autem valoribus loco m^4 & m^2, factaque reductione *per Lem. cit.* invenitur tandem æquatio

$$x^3 - 24d^2x - 48d^3 = 0$$

Quæ quidem eſt eadem, quam innuit Newtonus [*] in hoc eleganti problemate, cujus ipſe auctor eſt. Reſolvitur autem ultimo *per Prop.* 8. *vel* 9. *bujus.* Nam fiat $d = 1$, $p = 24$, $\frac{1}{3}p = 8$, & $\frac{1}{27}p^3 = 512$: item $q = 48$, $\frac{1}{2}q = 24$, & $\frac{1}{4}qq = 576$, invenitur per Cardani formulas

$$x = \sqrt[3]{32} + \sqrt[3]{16} = 2\sqrt[3]{4} + 2\sqrt[3]{2}.$$

CA-

* Arithm. Univerſ. Probl. Geom. XIV. Londini an. 1722.

CAPUT X.

De Æquationibus Biquadraticis & aliis altiorum graduum.

SEcundus terminus, radicales & fractiones ſupponuntur, ſi forte adſint, ab æquatione ſublatæ *per Propoſ.* 2. 8. *&* 9. *Cap.* 7.

PROPOSITIO I.

Æquationes biquadraticas ad cubicas reducere.

SIt æquatio generalis repræſentans omnes æquationes quarti gradus ſecundo termino carentes

$$x^4 + px^2 + qx + r = 0$$

Cum hæc concipi poſſit veluti orta ex duabus æquationibus ſecundi gradus, quas *componentes* vocabimus, nempe

$$x^2 + yx + f = 0, \ \& \ x^2 - yx + g = 0$$

Fiat ex earum multiplicatione æquatio æqualis priori $x^4 + px^2 + qx + r = 0$, ſecundo termino carens (ob terminos $+yx$ & $-yx$ contrario ſigno affectos) nimirum

$$\begin{aligned} x^4 &+ fx^2 - fyx + fg = 0 \\ &+ gx^2 + gyx \\ &- y^2x^2 \end{aligned}$$

Com-

Comparentur æquationis utriuſque termini, erunt tres æquationes

$$f+g-y^2=p,\ gy-fy=q,\ \&\ fg=r$$

Ex quibus efformari debet alia, in qua non occurrat ulla incognita præter y, quæ in utraque componente eſt ſecundi termini coefficiens.

Hinc quia $f+g-y^2=p$, erit quoque $f+g=p+y^2$, & multiplicando omnes terminos per y, fit

$$fy+gy=py+y^3$$

Sed habetur ex ſecunda æquatione $gy-fy=q$; ergo hæ duæ æquationes ſimul additæ efficiunt $2gy=py+y^3+q$. Si ſubtrahantur vero, fiunt $2fy=py+y^3-q$; unde eruuntur ipſarum f & g valores, ſcilicet

$$g=\frac{py+y^3+q}{2y},\ f=\frac{py+y^3-q}{2y}$$

$$fg=\frac{py+y\ \ -q}{2y}\times\frac{py+y^3+q}{2y}$$

$$fg=\frac{p^2y^2+2py^4+y^6-q^2}{4y^2}$$

Mult. per $4y^2$

$$4fgy^2=p^2y^2+2py^4+y^5-q^2$$

Erat autem $fg=r$, proinde ſi utrunque membrum multiplicetur per $4y^2$, erit $4fgy^2=4ry^2$, ideoque fiet

$$4ry^2=p^2y^2+2py^4+y^6-q^2$$

Et ordinata æquatione habetur æquatio generalis cubica, nimirum

$$\begin{aligned} y^6 + 2py^4 + p^2y^2 - q^2 &= 0 \\ - 4ry^2 & \end{aligned}$$

Quæ, licet ſexti gradus eſſe appareat, habetur tanquam cubica, cum omnes incognitæ termini ſint diviſibiles per 2, & dicitur *Cubica derivata* cui innititur æquationum omnium quarti gradus reſolutio, ut per exempla, quæ ſequuntur, planum fiet.

COROLL. I. *Cum æquatio hæc* Cubica derivata *ſit generalis, utpote orta ex æquatione illa generali* $x^4 + px^2 + qx + r = 0$, *quæ æquationes omnes quarti gradus repræſentare ſupponitur, hinc eſt, quod data quarti gradus æquatione ſpeciali, poterit ſtatim haberi ejus Cubica derivata, ſubſtituendo in cubica generali loco* p, q *&* r *valores æquationis particularis datæ. Sit data æquatio* $x^4 - 4x^2 - 8x + 35 = 0$, *erit* $p = -4$, $q = -8$, *&* $r = 35$. *Poſitis his valoribus in cubica generali, habetur cubica derivata particularis* $y^6 - 8y^4 - 124y^2 - 64 = 0$. *Similiter ſit data æquatio* $x^4 - 17x^2 - 20x - 6 = 0$, *erit* $p = -17$, $q = -20$, $r = -6$. *Unde Cubica derivata erit* $y^6 - 34y^4 + 313y^2 - 400 = 0$. *Quæ quidem ſemper conſiderari poterit ut ſimpliciter cubica, & tanquam mere cubica reſolvi, ſecundo termino ſublato. Nam ſi ponatur* $y^2 = z$, *fiet* $z^3 - 34z^2 + 313z - 400 = 0$, *& ob* $z = y^2$, *erit* $\sqrt{z} = y$.

COROLL. II. *Si in duabus illis componentibus* $x^2 + yx + f = 0$, *&* $x^2 - yx + g = 0$ *ponatur valor duarum indeterminatarum* f *&* g per hanc Propoſ. *inventus, oriuntur duæ æquationes*

$$x^2 + yx + \tfrac{1}{2}y^2 + \tfrac{1}{2}p - \frac{q}{2y} = 0$$

$$x^2 - yx + \tfrac{1}{2}y^2 + \tfrac{1}{2}p + \frac{q}{2y} = 0$$

Quæ quidem, cognito valore y, *fiunt omnino cognitæ & determinatæ ut patet.*

SCHOL. I. *Si in data æquatione particulari habeatur* $+p$, *tunc in utraque harum duarum ponitur* $+\tfrac{1}{2}p$. *Contra vero in utraque ponitur* $-\tfrac{1}{2}p$, *ſi æquatio particularis data habeat* $-p$.

SCHOL. II. *Si in data æquatione habeatur* $+q$ *tunc in una duarum, in qua ſcilicet habetur* $+yx$, *ponitur* $-\frac{q}{2y}$. *In altera, in qua eſt* $-yx$, *ponitur* $+\frac{q}{2y}$. *Contra vero ſi in data æquatione habeatur* $-q$, *ponendum eſt* $-\frac{q}{2y}$ *in illa, in qua habetur* $-yx$, *&* $+\frac{q}{2y}$ *in altera, ubi reperitur* $+yx$.

PROPOSITIO II.

Æquationes quarti gradus reſolvere.

ÆQuatio data reducatur ad cubicam *per Cor.* 1. *Prop. præc.* & ex cubica derivata extrahatur radix *per Prop.* 4. *vel* 5. *Cap.* 9. ſi ſit rationalis, vel *per Propoſ.* 8. *aut* 9. ſi ſit irrationalis. Radicis hujus cubicæ valor, tum etiam valores *p* & *q* æquationis datæ ponantur in duabus

 illis

illis indeterminatis ſecundi gradus, quæ habentur in *Coroll. 2. Prop. præc.*, facile erit obtinere quatuor æquationis datæ radices, quemadmodum exempla rem illuſtrabunt.

I. Sit æquatio, qua Carteſius (*a*) ipſe utitur, $x^4 - 17x^2 - 20x - 6 = 0$, erit (facta comparatione cum formula generali $x^4 + px^2 + qx + r = 0$) $p = -17$, $q = -20$, $r = -6$, ejusque cubica derivata $y^6 - 34y^4 + 313y^2 - 400 = 0$ *per Coroll.* 1. *Prop. præc.* Inveniatur valor y *per Prop.* 4. *vel* 5. *Cap.* 9. tollendo prius ſecundum terminum. Vel brevius reperiri poteſt, dividendo æquationem ipſam per aliquod quadratum, quod inter diviſores ultimi termini exiſtat, ex. gr. per $yy - 16$. Sic enim diviſio exacte ſuccedit; unde $yy = 16$, & $y = 4$.

Hoc valore una cum valoribus p & q poſito in duabus æquationibus $x^2 + yx + \frac{1}{2}y^2 + \frac{1}{2}p - \frac{q}{2y} = 0$, & $x^2 - yx + \frac{1}{2}y^2 + \frac{1}{2}p + \frac{q}{2y} = 0$, habentur duæ æquationes ſecundi gradus $x^2 + 4x + 2 = 0$, & $x^2 - 4x - 3 = 0$ *per Coroll. 2. Prop. præc.* in quibus apparent ſigna + & —, ut docuimus *in Schol.* 1. & 2. *Prop. cit.* earumque reſolutio obtinetur *per Prop.* 1. & 3. *Cap.* 8. Radices enim prioris ambæ negativæ ſunt $-2 + \sqrt{2}$, & $-2 - \sqrt{2}$; alterius vero affirmativæ $2 + \sqrt{7}$, & $2 - \sqrt{7}$.

II. Sit æquatio $x^4 - 12x^2 + 12x - 3 = 0$: comparatis terminis cum formula generali, erit $p = -12$, $q = 12$, & $r = -3$; ejuſque cubica derivata $y^6 - 24y^4 + 156y^2 - 144 = 0$ *per Coroll.* 1. *Prop. præc.* Cujus radix invenitur *per Prop.* 4. *vel* 5. *Cap.* 9. Vel brevius dividendo

(*a*) Geometr. lib. III. edit. 3. pag. m. 80.

dendo æquationem per aliquem ultimi termini diviſorem v. g. per $yy - 12$, per quem diviſio exacte ſuccedit. Eſt ergo $yy = 12$, & $y = 2\sqrt{3}$.

Ponatur hic valor ſimul & valores p & q in duabus indeterminatis ſecundi gradus, ut factum eſt ſupra in primo exemplo, erunt duæ ſecundi gradus æquationes $x^2 + 2x\sqrt{3} - \sqrt{3} = 0$, & $x^2 - 2x\sqrt{3} + \sqrt{3} = 0$ *per Coroll.* 2. & *Schol.* 1. & 2. *Prop. præc.* Quarum radices facile obtinentur *per Prop.* 1. *Cap.* 8. addendo utrique quadratum quantitatis cognitæ ſecundi termini diviſæ per 2, nempe addendo $+ 3$ (nam $\frac{2\sqrt{3}}{2} = \sqrt{3}$, cujus quadratum $= 3$)

Prioris ergo radices ſunt $\sqrt{3+\sqrt{3}} - \sqrt{3}$, & $-\sqrt{3+\sqrt{3}} - \sqrt{3}$. Poſterioris vero ſunt $\sqrt{3-\sqrt{3}} + \sqrt{3}$, & $-\sqrt{3-\sqrt{3}} + \sqrt{3}$.

III. Sit æquatio $x^4 - 6x^2 + 12x - 10 = 0$. Comparatis terminis cum formulæ generalis terminis, erit $p = -6$, $q = 12$ & $r = -10$, & ejus cubica derivata $y^6 - 12y^4 + 76y^2 - 144 = 0$, ſeu $z^3 - 12z^2 + 76z - 144 = 0$ *per Coroll.* 1. *Prop. præc.* Hæc autem cum nullum diviſorem admittat, indicio eſt, non habere valorem rationalem. Inveniatur ergo valor ejuſdem *per Propoſ.* 8. *Cap.* 9. Sed prius auferatur ab æquatione ſecundus terminus hoc pacto. Fiat $z - 4 = v$, erit $z = v + 4$, factaque poteſtatum ſubſtitutione, habetur transformata ſecundo termino carens, ſcilicet

$$\begin{array}{r|rrrr}
z^3 & = v^3 + 12v^2 & + 48v & + 64 \\
-12z^2 & -12v^2 & -96v & -192 \\
+76z & & +76v & +304 \\
-144 & & & -144 \\
\hline
\text{Summa} \quad v^3 & * & +28v & +32 = 0
\end{array}$$

Ex hac æquatione eruitur valor v *per Schol. Propoſ.* 8. *Cap.* 9. nempe

$$v = \sqrt[3]{-16 + \sqrt{\tfrac{28864}{27}}} - \sqrt[3]{16 + \sqrt{\tfrac{28864}{27}}}.$$

Sed poſita fuit $z - 4 = v$, proinde erit

$$z = 4 + \sqrt[3]{-16 + \sqrt{\tfrac{28864}{27}}} - \sqrt[3]{16 + \sqrt{\tfrac{28864}{27}}}$$

Erat autem $y^2 = z$, unde $y = \sqrt{z}$; atque hinc

$$y = 2 + \sqrt{\sqrt[3]{-16 + \sqrt{\tfrac{28864}{27}}} - \sqrt[3]{16 + \sqrt{\tfrac{28864}{27}}}}$$

Hoc valore ipſius y ſimul cum valoribus p & q ſurrogatis in duabus illis æquationibus ſecundi gradus, ut in primo & ſecundo exemplo factum eſt, hæ determinantur *per Coroll.* 2. *Prop. præc.* & quatuor, quæ inde eruuntur, radices dant radices æquationis propoſitæ. Hoc autem cum calculum valde prolixum & permoleſtum exigat, neceſſe non erit ultra pergere: nam problemata, quæ ab æquationibus hujus generis dependent, ad Solidorum Geometriam amandantur.

CORÒLL. *Valor incognitæ* y *ex cubica derivata inventus vel rationalis eſt, ut in primo exemplo; vel radices*

quadra-

quadraticas continet, ut in ſecundo; vel denique eſt irrationalis radicales cubicas involvens, ut in tertio. In primo & ſecundo caſu æquatio quarti gradus dicitur plana, *& in duas ſecundi gradus diviſibilis eſt, ut vidimus. In tertio caſu æquatio vere & proprie quarti gradus dicitur, & a nonnullis* affectionem cubicam *continere. Problema autem, ad quod ſolvendum ordinatur, dicitur* ſolidum, *quod non per regulam & circinum, ſed per aliquam ſectionem Conicam, cujus ope radices exhibentur, conſtrui tantummodo poteſt.*

PROPOSITIO III.

An biquadraticæ, quæ quatuor radicibus imaginariis conſtant, reſolvi poſſint.

Cum certum ſit, biquadraticas, quæ duas radices reales & duas imaginarias habent, reſolvi poſſe, ex eo quod tunc earum cubica derivata unam realem, & duas imaginarias radices contineat, adeoque ſolubilis ſit *per Propoſ.* 4. *Cap.* 9.; cum tamen biquadratica quatuor imaginariis radicibus conſtat, tunc omnes ejus cubicæ derivatæ radices reales fiunt, proinde oritur dubium, an in eo caſu reſolvi valeat.

I. Sit igitur æquatio $x^4 + x^2 + 2x + 6 = 0$. Comparentur hujus termini cum terminis formulæ generalis $x^4 + px^2 + qx + r = 0$, erit $p = 1$, $q = 2$, $r = 6$, & ejus cubica derivata $y^6 + 2y^4 - 23y^2 - 4 = 0$ *per Cor.* 1. *Prop.* 1. *hujus*. Hæc ſi dividatur per $yy - 4$, nihil remanet, proinde $y = 2$: quo valore una cum valoribus p & q in dua-

bus

bus componentibus *per Cor. 2. Prop. cit.* fiunt $x^2 - 2x + 3 = 0$, & $x^2 + 2x + 2 = 0$, earumque radices, ut ex ſignis patet, ſunt imaginariæ; nempe $1 + \sqrt{-2}, 1 - \sqrt{-2}$, & $-1 + \sqrt{-1}, -1 - \sqrt{-1}$.

II. Sit æquatio $x^4 + 70x^2 - 3744x + 27993 = 0$, comparatis terminis cum terminis formulæ generalis ut ſupra, invenitur ejus cubica derivata *per Coroll. 1. Prop. 1. hujus* $y^6 + 140y^4 - 107072y^2 - 14017536 = 0$, & dividendo illam per $yy - 324$, cum diviſio exacte fiat, erit $yy = 324$, & $y = 18$; quo valore in duabus componentibus poſito *per Coroll. 2. Prop. cit.* fiunt $x^2 - 18x + 93 = 0$, & $x^2 + 18x + 301 = 0$, ex quibus quatuor radices imaginariæ prodeunt, $9 + \sqrt{-12}, 9 - \sqrt{-12}, -9 + \sqrt{-220}, -9 - \sqrt{-220}$.

Coroll. I. *Atque hinc patet, biquadraticas, quæ quatuor radicibus imaginariis conſtant, ſortiri poſſe cubicam derivatam, quæ radicem aliquam realem & rationalem contineat, quæ quidem vel per aliquem diviſorem, vel per extractionem radicis ex binomio cubico obtineatur, tunc vero tam ipſam quam biquadraticam optime poſſe reſolvi. Quanquam biquadratica ejuſmodi ad problematis ſolutionem, ad quod erat ordinata, inepta ſit, cum nulla radix realis ad illud conſtruendum aſſignari valeat, ut Carteſius advertit.*

Coroll. II. *At generatim ſi cubica derivata talis ſit, ut tres radices reales irrationales contineat, ita ut* caſum irreductibilem *involvat, de quo* in Schol. 3. Prop. 9. Cap. 9. *tunc neque ipſa, neque biquadratica ſolvi poterunt, cum omnis æquationum quarti gradus reſolutio a cubicæ derivatæ ſolutione maxime pendeat, eique tanquam fundamento innitatur.*

PRO-

PROPOSITIO IV.

Æquationes quarti gradus alia ratione reſolvere.

ALia quoque methodus, æquationes biquadraticas reſolvendi, & illas in duas planas ſecundi gradus dividendi, ſi diviſibiles ſint, excogitata fuit a Cl. Huddenio; quæ inſuper id habet commodi, ut, ſi adſit ſecundus terminus, nihil obſtet.

Sit æquatio generalis repræſentans omnes æquationes quarti gradus, etiam ſecundo termino præditas

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Dividatur in duas partes, ita ut altera contineat $x^4 + px^3$, altera $x^2 + rx + s$, utraque ſcilicet parium dimenſionum poteſtates x^4 & x^2, erit

$$x^4 + px^3 = - qx^2 - rx - s$$

Ut prima pars quadratum fiat, ponatur radix $x^2 + \frac{1}{2}px + \frac{1}{2}y$, ejus quadratum erit

$$\begin{aligned} x^4 + px^3 + \tfrac{1}{4}p^2x^2 &+ \tfrac{1}{2}pyx + \tfrac{1}{4}yy \\ + yx^2 & \end{aligned}$$

Quod quidem alteri quoque æquationis parti addatur, omiſſis $x^4 + px^3$, cui æqualis ſupponitur $- qx^2 - rx - s$; ita ut loco $x^4 + px^3 = - qx^2 - rx - s$ habeatur

$$\begin{aligned} x^4 + px^3 + \tfrac{1}{4}p^2x^2 + \tfrac{1}{2}pyx + \tfrac{1}{4}yy &= - qx^2 - rx - s \\ yx^2 \qquad\qquad\qquad & \quad\ \tfrac{1}{4}p^2x^2 + \tfrac{1}{2}pyx + \tfrac{1}{4}yy \\ & \quad\ yx^2 \end{aligned}$$

Cum igitur prima pars ſit quadratum, altera quoque pars quadrato æquari debet. Ponatur hujus radix eſſe $vx + z$, ejus quadratum erit $vvx^2 + 2vzx + zz$; utque habeantur valores v & z, fiat terminorum comparatio, erit

$$vv = \tfrac{1}{4}pp - q + y, \ \& \ v = \sqrt{\tfrac{1}{4}pp - q + y}$$

$$2vz = \tfrac{1}{2}py - r, \ \& \ z = \frac{\frac{1}{2}py - r}{2v}, \text{hoc eſt}$$

$z = \dfrac{\frac{1}{2}py - r}{2\sqrt{\frac{1}{4}pp - q + y}}$. Proinde radix $vx + z$, erit

$$\sqrt{\tfrac{1}{4}pp - q + y} \times x + \frac{\frac{1}{2}py - r}{2\sqrt{\frac{1}{4}pp - q + y}}, \text{cujus quadratum}$$

$$\overline{\tfrac{1}{4}pp - q + y} \times xx + \overline{\tfrac{1}{2}py - r} \times x + \frac{\frac{1}{4}ppyy - pry + rr}{pp - 4q + 4y}$$

Comparando jam hujus quadrati terminos cum terminis quadrati ſecundæ partis præcedentis æquationis, deſtruuntur termini ſimiles, & remanet

$$\frac{\frac{1}{4}ppyy - pry + rr}{pp - 4q + 4y} = \tfrac{1}{4}yy - s$$

Factaque multiplicatione, & ordinata æquatione, oritur cubica generalis

$$\begin{array}{l} y^3 - qyy - 4sy - pps = 0 \\ \qquad\qquad\ \ + pry + 4qs \\ \qquad\qquad\qquad\quad\ - rr \end{array}$$

Cujus valoribus p, q, r, s per coefficientes æquationis ſpecialis

cialis datæ determinatis, quæritur valor ipſius y. Quo invento, ſi rationalis ſit, poterunt ejuſdem ope determinari duæ ſequentes æquationes ſecundi gradus, quæ ex æqualium quadratorum radicibus componuntur, & per illas æquatio propoſita dividi, ut, quæ inferius ſequuntur exempla, rem oſtendunt.

$$x^2 + \tfrac{1}{2}px + \sqrt{\tfrac{1}{4}pp - q + y} \times x + \tfrac{1}{2}y + \frac{\tfrac{1}{2}py - r}{2\sqrt{\tfrac{1}{4}pp - q + y}} = 0$$

$$x^2 + \tfrac{1}{2}px - \sqrt{\tfrac{1}{4}pp - q + y} \times x + \tfrac{1}{2}y - \frac{\tfrac{1}{2}py - r}{2\sqrt{\tfrac{1}{4}pp - q + y}} = 0$$

I. Data ſit æquatio $x^4 - x^3 - 3x^2 + x + 2 = 0$, erit $p = -1$, $q = -3$, $r = 1$, $s = 2$, quibus valoribus in cubica generali poſitis, fit $y^3 + 3y^2 - 9y - 27 = 0$, quæ ſi dividatur per $y + 3$, nihil remanet; eſt ergo $y = -3$, qui valor, cum ſit rationalis, indicio eſt, æquationem datam in duas ſecundi gradus eſſe diviſibilem. Ponatur igitur hic valor una cum valoribus p, q, r in duabus præcedentibus ſecundi gradus, eruitur ex prima $x^2 - 1 = 0$, ex altera $x^2 - 1x - 2 = 0$, & per utranque æquatio data dividi poteſt, ut patet.

II. Sit Carteſii æquatio a nobis allata *in Propoſ. præc.* $x^4 - 17x^2 - 20x - 6 = 0$. Erit $p = 0$, $q = -17$, $r = -20$, $s = -6$; & per hos valores cubica generali determinata, invenitur $y^3 + 17y^2 + 24y + 8 = 0$, quæ ſi dividatur per $x + 1$, nihil remanet. Eſt ergo $y = -1$, & hoc valore in duabus ſecundi gradus ſubſtituto, inveniuntur $x^2 + 4x + 2 = 0$, & $x^2 - 4x - 3 = 0$, omnino ut antea per Carteſii regulam.

COROL. *Si valor ipſius* y *ex cubica inventus non ſit rationalis, æquatio propoſita quarti gradus non erit diviſibilis in duas ſecundi gradus, ſed problema erit ſolidum.*

SCHOL. *Cum præclaræ hujus methodi artificium* Huddenius *nobis occultum reliquerit, illud* Antonii de Monforte *ſolertia detexit* ([a]). *Quod & nos, paucis mutatis, hoc loco referre in laudem Cl. Viri de Matheſi optime meriti, æquum duximus.*

PROPOSITIO V.

Æquationem biquadraticam puram, vel ſecundo & quarto termino carentem ſolvere.

I. Sit æquatio biquadratica pura $x^4 = q$, vel $x^4 = -q$, extrahatur primo radix quadrata, erit $x^2 = \sqrt{q}$, vel $x^2 = \sqrt{-q}$. Deinde iterum extrahatur eadem radix, & habebitur $x = \sqrt{\sqrt{q}}$, vel $x = \sqrt{\sqrt{-q}}$. Sit ex. gr. $x^4 = 50$, erit $x^2 = \sqrt{50} = 5\sqrt{2}$, extractaque rurſus radice, erit $x = \sqrt{5\sqrt{2}}$.

II. Si in æquatione biquadratica, præter ſecundum terminum, deſit etiam quartus (ſæpe fit, ut auferendo ſecundum terminum, quartus quoque evaneſcat) ut $x^4 - 6x^2 - 4 = 0$, facile erit eam reſolvere ut derivativam ſecundi gradus *per Prop.* 4. *Cap.* 8.

COROL. *Omnes ergo biquadraticæ, quæ primum & ultimum terminum tantum habent, vel quæ termino ſecundo,*

(a) Tract. de Probl. determinat. Neap. 1690.

do, & quarto carent, ab omni affectione cubica ſunt immunes, & in duas ſecundi gradus dividi poſſunt.

SCHOL. Raphael Bombelli (a) *Bononienſis anno* 1579. *æquationes biquadraticas reſolvendi methodum in ſua Algebra tradidit, ſed* a Ludovico de Ferrariis *eam accepiſſe ferunt. Quicquid ſit, Italorum hominum ingenium & gloria fuit hucuſque progredi* (b). *Poſt* Vietam Carteſius *nova ſua methodo mirum quantum illam illuſtravit, quem deinde ſecuti ſunt ducem, quotquot de Analyſi ſcripſerunt. Ceterum majores noſtri ultra æquationes biquadraticas non ſunt progreſſi, ſummam rei difficultatem proſpicientes. Recentiores aliqui generalem æquationes quaſcunque reſolvendi methodum tradideruut; non advertentes fortaſſe eam methodum inutilem jure cenſeri, quæ immenſum (quod ipſi fatentur) & inexplicabilem fere calculum poſtulat. Nos duntaxat de reductione æquationum quarti, quinti, & ſexti gradus paulo poſt agemus: quæ ſi reducibiles ſint, facile poſſunt per ea, quæ de cubicis & biquadraticis diximus reſolvi. Sin autem reduci non poterunt, ſaltem conſtabit, problema, ad quod illæ ordinantur, eſſe ſolidum, & nonniſi per ſectiones Conicas reſolvendum. Sed tyronum ſtudio obſequentes, nonnulla quarti gradus problemata ad eorum ingenii exercitationem præmittimus.*

PRO-

(a) Algebra lib. 2. pag. 353. Wallisii Algeb. cap. IV. pag. 229. (b) Histoire de l'Acad. Royale des ſcien. an. 1705. pag. m. 103.

PROBLEMATA QUARTI GRADUS

PROBLEMA I.

Numerum datum in duos ita dividere, ut eorum quadrata inter ſe multiplicata producant numerum æqualem numero dato c.

ESto numerus datus dividendus $= 2a$, & partium differentia $= 2x$, erit pars major $a + x$, minor vero $a - x$ *per Theor.* 3. *Prop.* 3. *Cap.* 5. Et ex conditione problematis habetur

$$\overline{a+x}^2 \times \overline{a-x}^2 = c$$
$$x^4 - 2a^2x^2 + a^4 = c$$

Brevitatis gratia, ut quantitates a & c determinentur, pono $2a = 14$, & $c = 2304$, erit æquatio, & quidem derivativa ſecundi gradus, facile reſolvenda *per Prop.* 4. *Cap.* 8.

$$x^4 - 98x^2 + 2401 = 2304$$
$$x^4 - 98x^2 = -97$$
$$\textit{adde} \quad 2401 \qquad 2401$$
$$x^4 - 98x^2 + 2401 = 2304$$

Extr. rad. $x^2 - 49 = \pm\sqrt{2304} = \pm 48$

$$x^2 = 49 - 48 = \sqrt{1}$$
$$x = 1$$

Unde

Unde $a+x=8$, & $a-x=6$, quorum quadratis invicem multiplicatis æquatur $c=2304$.

PROBLEMA II.

Invenire quatuor numeros in continua proportione Arithmetica, qui inter ſe multiplicati faciant numerum a.

Pone numerum datum $a=100$, & terminorum differentiam $=d$. Sit primus terminus x, erit ſecundus $x+d$, tertius $x+2d$, & quartus $x+3d$ *per Sch.* 1. *Propoſ.* 3. *Cap.* 5. quibus invicem multiplicatis, habetur æquatio

$$x^4+6dx^3+11d^2x^2+6d^3x-100=0$$

Inventis autem *per Prop.* 9. *Cap.* 1. ultimi termini diviſoribus 1, 2, 4, 5 &c. tentetur diviſio per $x\pm1$, $x\pm2$, $x\pm4$ &c. quæ non ſuccedit ſine reſiduo. Tollatur ſecundus terminus æquationis *per Prop.* 5. *Cap.* 7. Quamobrem fiat $x=z-\frac{3}{2}d$, æquatio transformabitur in ſequentem derivativam ſecundi gradus, quæ ſecundo & quarto termino caret, nempe

$$z^4 * -\tfrac{5}{2}d^2z^2 * +\tfrac{9}{16}d^4-100=0$$

Proinde ſi ponatur $d=1$, erit

$$z^4-\tfrac{5}{2}z^2=99\tfrac{7}{16}$$

adde $\quad\frac{25}{16} \qquad \frac{25}{16}$

$$z^4-\tfrac{5}{2}z^2+\tfrac{25}{16}=101$$

Extr. rad.

$$z^2-\tfrac{5}{4}=\sqrt{101}$$

Extr.

$$\text{Extr. rad.} \quad z^2 = \tfrac{5}{4} + \sqrt{101}$$
$$z = \sqrt{\tfrac{5}{4} + \sqrt{101}}$$

Sed poſita fuit $x = z - \frac{3}{2}d$, erit ergo primus terminus quæſitus $x = \sqrt{\frac{5}{4} + \sqrt{101}} - \frac{3}{2}$, ſecundus $x + d = \sqrt{\frac{5}{4} + \sqrt{101}} - \frac{1}{2}$, tertius $x + 2d = \sqrt{\frac{5}{4} + \sqrt{101}} + \frac{1}{2}$, quartus $x + 3d = \sqrt{\frac{5}{4} + \sqrt{101}} + \frac{3}{2}$. Factum ex primo & quarto eſt $-1 + \sqrt{101}$, factum vero ex ſecundo & tertio eſt $+1 + \sqrt{101}$ *per num.* 4. *Propoſ.* 12. *Cap.* 4. Atque hinc $\overline{-1 + \sqrt{101}} \times \overline{+1 + \sqrt{101}} = 100$.

Sit iterum $a = 100$, $d = 2$, æquatio ſuperior fiet $z^4 - 10z^2 - 91 = 0$, ſeu

$$z^4 - 10z^2 = 91$$
$$\text{adde} \quad 25 \quad 25$$
$$z^4 - 10z^2 + 25 = 116$$
$$\text{Extr. rad.} \quad z^2 - 5 = \sqrt{116}$$
$$z^2 = 5 + \sqrt{116}$$
$$z = \sqrt{5 + \sqrt{116}}$$

Sed $x = z - \frac{3}{2}d$, ergo primus terminus proportionalis erit $x = \sqrt{5 + \sqrt{116}} - 3$, ſecundus $x + d = \sqrt{5 + \sqrt{116}} - 1$, tertius $x + 2d = \sqrt{5 + \sqrt{116}} + 1$, quartus $x + 3d = \sqrt{5 + \sqrt{116}} + 3$. Proinde factum ex primo & quarto

to eſt $-4+\sqrt{116}$, factum vero ex ſecundo & tertio $+4+\sqrt{116}$ *per Propoſ. citat.*, adeoque $\overline{-4+\sqrt{116}} \times \overline{+4+\sqrt{116}} = 116 - 16 = 100$.

PROBLEMA III.

Tres numeros invenire, quorum quadrata ſint harmonice proportionalia.

Sit quæſitorum primus 1, ſecundus x, tertius $x+1$, eorumque quadrata 1, xx, x^2+2x+1: quæ cum ſint ex hypotheſi harmonice proportionalia, erit maximum ad minimum, ut differentia maximi & medii ad differentiam medii & minimi, nimirum

$$x^2+2x+1 \cdot 1 :: 2x+1 \cdot xx-1$$

Et multiplicatis mediis & extremis, habetur

$$x^4+2x^3-2x-1=2x+1$$
$$x^4+2x^3-4x-2=0$$

Quæ quidem æquatio cum per nullum binomium diviſibilis ſit, tentanda eſt ſolutio *per Propoſ.* 4. *hujus.*

Sit igitur $p=2$, $q=0$, $r=-4$, $s=-2$, poſitiſque his valoribus in cubica generali, oritur $y^3-8=0$, hoc eſt $y=2$. Quo valore una cum valoribus p, r, s ſubſtituto in duabus ſecundi gradus æquationibus, fiunt

$$x^2 \begin{array}{l}+1x\\+\sqrt{3}\times x\end{array} +1+\frac{3}{\sqrt{3}}=0, \&\ x^2 \begin{array}{l}+1x\\-\sqrt{3}\times x\end{array} +1-\frac{3}{\sqrt{3}}=0$$

Cum autem ex neutra habeatur radix, quæ numeris explicari poſſit, ut evidens eſt, aliam viam, quam Jacobus de Billy (*a*) innuit ex Diophanti methodo, ingredimur, & eſt hujuſmodi.

Inveniantur duo numeri quadrati tales, ut eorum differentia addita majori faciat quadratum. Ponantur, ut ſupra, quadrata $x^2 + 2x + 1$, & xx, quorum differentia $2x + 1$ addita majori, fit $x^2 + 4x + 2$. Cum autem hæc ſumma debeat æquari quadrato, ſi pro ejus latere ſumatur $x - 2$, erit $x^2 - 4x + 4 = x^2 + 4x + 2$; & ſublatis utrinque terminis ſimilibus, habetur $x = \frac{1}{4}$, quo valore in duobus quadratis aſſumptis poſito, erunt $\frac{25}{16}$ & $\frac{1}{16}$, ſeu in integris 25 & 1, quorum differentia 24 majori quadrato addita facit quadratum 49, ut patet. Jam igitur numeri quadrati harmonice proportionales ab initio poſiti 1, xx, $x^2 + 2x + 1$ erunt 1, xx, $25xx$; adeoque $25xx . 1 :: 24xx . xx - 1$, & multiplicatis tum mediis, tum extremis, oritur $25x^4 - 25x^2 = 24x^4$, ſeu $25x^2 = 49$, hoc eſt $x^2 = \frac{49}{25}$, & $25x^2 = \frac{1225}{25}$, proinde numeri quæſiti ſunt 1, $\frac{49}{25}$, & $\frac{1225}{25}$, vel in integris 25, 49, 1225.

COROLL. *Si pro quadrati latere* $x - 2$, *ponatur* $x - 3$, *vel* $x - 4$ *&c. invenientur alii atque alii in infinitum tres numeri quadrati harmonice proportionales, ut ſupra.*

SCHOL. *Duo hic notent tyrones: 1.° Hujus problematis ſolutionem ab ea conditione neceſſario pendere, ut duorum quadratorum, quæ aſſumuntur, differentia majori quadrato addita quadratum efficiat: 2.° Non eſſe omnino deſperandum de ſolutione problematis, quæ primo impoſſibilis videatur. Nam ſi nequit per unam methodum, per aliam ſæpe ſatis commode expeditur.*

PROBLE-

(*a*) Diophanti Redivivi p. 2. Probl. XVI.

PROBLEMA IV.

In rectangulo datis areæ & diametri aggregatto, & differentia laterum, invenire latera, diametrum, & aream. Fig. 8.

Sit areæ & diametri ſumma $= a$, differentia laterum $= 2d$, quæſitorum eorundem laterum ſumma $= 2x$, erit latus minus $AB = x - d$, majus vero $BC = x + d$ *per Theor.* 3. *Prop.* 3. *Cap.* 5. & ex eorum facto innoteſcit rectanguli BD area $= x^2 - d^2$, quæ ſi auferatur ex ſumma data $= a$, prodit diameter $AC = a - x^2 + d^2$.

Cum igitur ſit $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ *per Propoſ.* 47. *lib.* 1. *Eucl.*, erit in terminis analyticis

$$\overline{a - x^2 + d}^2 = \overline{x + d}^2 + \overline{x - d}^2$$

$$a^2 - 2ax^2 + 2ad^2 + x^4 - 2d^2 x^2 + d^4 = 2x^2 + 2d^2$$

$$\begin{aligned} x^4 - 2d\ \ x^2 &= 2d^2 - 2ad^2 - d^4 - a^2 \\ -\ 2ax^2 & \\ -\ 2x^2 & \end{aligned}$$

Et ſi ponatur $a = 58$, $d = 1$, æquatio determinabitur, nempe

$$x^4 - 120x^2 = -3479$$

Prop. 4. *Cap.* 8. $\quad 3600 \qquad 3600$

$$x^4 - 120x^2 + 3600 = 121$$

Extr. rad. $\quad x^2 - 60 = \pm \sqrt{121}$

$$x^2 = 60 \mp 11 = 49$$

Extr. rad. $\quad x = 7$

Erit ergo latus minus $x - d = 6$, majus $x + d = 8$, & area rectanguli $= 48$, quæ detracta ex aggregato areæ & diametri a $(= 58)$ relinquit diametrum $= 10$, ut evidens eſt.

PROBLEMA V.

Data ſumma laterum trianguli, dataque ratione ſummæ quadratorum, quæ fiunt a lateribus, ad aream ejuſdem trianguli, reperire latera. Fig. 9.

Pone ſummam laterum datam $= 2a$, atque hinc ſume baſim $AB = b$, erit aggregatum reliquorum $AC + CB = 2a - b$. Eſto ex illis unum $AC = x$, erit aliud $2a - b - x$. Ratio vero ſummæ quadratorum ad trianguli aream ponatur ut m ad n, ſive 8 ad 1.

Ut inveniatur trianguli ipſius area, ſubtrahe ex ſemiſumma laterum $= a$ ſingula trianguli ejuſdem latera, erunt tres differentiæ $a - b$, $a - x$, $-a + b + x$, ex quarum facto ducto in ſemiſummam laterum obtinetur trianguli area per ea, quæ diximus *in Problem.* 4. *Cap.* 8. nempe

$\sqrt{-a^4 + 2a^3 b + 2a^3 x - 3a^2 bx - a^2 x^2 - a^2 b^2 + ab^2 x + abx^2}$, quam voco g.

Summa vero quadratorum ex lateribus ejuſdem trianguli erit

$2x^2 + 2bx - 4ax + 2b^2 - 4ab + 4a^2$, quam voco f.

Eſt autem ex conditione problematis

$$f \cdot g :: m \cdot n$$

Th.6.Prop.3.Cap.5. $f^2 \cdot g^2 :: m^2 \cdot n^2$

Theor. 4. Prop. cit. $f^2 \times n^2 = g^2 \times m^2$

Poſi-

Poſitis proinde valoribus m^2, n^2, f^2 & g^2, factaque de more reductione, oritur æquatio

$$\begin{aligned} x^4 - 4ax^3 + 24a^2 x^2 - 40a^3 x - 40a^3 b &= 0 \\ + 2bx^3 - 24abx^2 + 60a^2 bx + 24a^2 b^2 & \\ + 3b^2 x^2 - 24ab^2 x - 4ab^3 & \\ + 2b^3 x + 20a^4 & \\ + b^4 & \end{aligned}$$

Jam ſi ponatur $a = 6$, & $b = 3$, prodit eadem æquatio determinata $x^4 - 18x^3 + 459x^2 - 3402x + 7209 = 0$. Hæc autem cum per nullum binomium dividi poſſit ſine reſiduo, quærenda eſt, ſublato ſecundo termino, ejus ſolutio *per Prop.* 2. *vel* 3. *hujus*.

Quod ſi manentibus, ut prius, $a = 6$, $b = 3$, fiat $m \cdot n = 8\frac{1}{3} : 1$, ſeu $m \cdot n = 25 \cdot 3$, eademque methodo procedatur, invenietur æquatio determinata

$$\begin{array}{ccccc} x^4 - 18x^3 & + 483\tfrac{1}{2}x^2 & - 3622\tfrac{1}{2}x & + 7650 = 0 \\ 1. \quad 4. & 16. & 64. & 256. \end{array}$$

$$z^4 - 72z^3 + 7736z^2 - 231840z + 1958400 = 0$$

Quæ ſi dividatur per $z - 16$, nihil remanet; eſt ergo $z = 16$. Sed ob progreſſionem Geometricam quadruplam, qua multiplicatæ ſunt radices prioris æquationis, erit $4x = z$, ſeu $4x = 16$, & $x = 4$. Proinde latus quæſitum $AC = 4$.

PRO-

PROBLEMA VI.

Dato quadrato AC, *& recta* M *producere latus* DC *in* E, *ita ut ex angulo* A *ducta* AE, *pars intercepta* EF *æqualis ſit datæ* M. Fig. 24.

SIt AB, vel $BC = a$, $BF = x$, erit $FC = a - x$. Sit M (ex hypotheſi $= FE$) $= c$. Ob triangula ſimilia CFE, & AFB, erit $CF\ (a - x) \cdot FE\ (c) :: FB\ (x) \cdot AF\ \frac{cx}{a - x}$. Et ob triangulum ABF rectangulum eſt $\overline{AF}^2 = \overline{AB}^2 + \overline{BF}^2$, hoc eſt $\frac{ccxx}{aa - 2ax + xx} = aa + xx$; atque hinc oritur

$$\begin{array}{l} x^4 - 2ax^3 + 2aax^2 - 2a^3x + a^4 = 0 \\ \qquad\qquad\quad - ccx^2 \end{array}$$

Ut tollatur ſecundus terminus, fiat $x - \frac{1}{2}a = z$. oritur

$$\begin{array}{l} z^4 * + \frac{1}{2}aaz^2 - a^3z + \frac{5}{16}a^4 = 0 \\ \qquad\quad - ccz^2 - accz - \frac{1}{4}aacc \end{array}$$

Invenienda modo eſt ejus cubica derivata *per Prop.* I. *hujus ejusque Coroll.* proinde neceſſe eſt noſſe, an termini hujus biquadraticæ ſint cum ſigno +, vel —. Quod ex ipſis dati problematis conditionibus expiſcari debemus; hoc eſt videndum, an $a > c$, vel contra $c > a$. Certum eſt $FE\ (= M)$ eſſe $> AB$, vel BC; adeoque $c > a$, item $cc > aa$, & multo magis $cc > \frac{1}{2}aa$. Igitur tertius æquationis datæ terminus eſt negativus, nempe $\frac{1}{2}aa - cc = -p$. Similiter $-a^3 - acc = -q$. Demum quia $cc > aa$, erit

cc>

$cc > \frac{5}{4}aa$, & multiplicando utrinque per aa, erit $aacc > \frac{5}{4}a^4$, & $\frac{1}{4}aacc > \frac{5}{16}a^4$, unde ultimus terminus eſt negativus, nempe $\frac{5}{16}a^4 - \frac{1}{4}aacc = -r$. Omnes ergo biquadraticæ termini ſunt cum ſigno $-$. Nimirum $2p = aa - 2cc$, $p^2 - 4r = -a^4 + c^4$, $-q^2 = -a^6 - 2a^4cc - aac^4$. Quibus in cubica generali derivata poſitis, habetur *per Prop. 11.*

$$\begin{array}{l} y^6 + aay^4 - a^4y^2 - a^6 = 0 \\ \quad\; - 2ccy^4 + c^4y^2 - 2a^4cc \\ \qquad\qquad\qquad\qquad - aac^4 \end{array}$$

Jam videndum, an hæc diviſibilis ſit per aliquem ultimi termini diviſorem ſine reſiduo. Diviſores ſunt a, aa, $aa + cc$, $a^3 + acc$ *per Prop.* 9. *Cap.* 1. Diviſio tentari poteſt per $yy - aa$, vel $yy - aa - cc$. Et quidem dividendo per hunc ſecundum, nihil remanet. Quotus autem eſt $y^4 - ccy^2 + 2aay^2 + a^4 + aacc = 0$, unde patet cubicam derivatam conflari ex duabus æquationibus, una ſecundi, altera quarti gradus. At nobis ſatis eſt æquatio ſecundi gradus $yy - aa - cc = 0$, unde habemus valorem ipſius y, hoc eſt $y = \pm\sqrt{aa + cc}$, qui valor ſubſtituendus eſt in his duabus indeterminatis ſecundi gradus *per Prop.* 1. *hujus* cum notis & obſervationibus *Schol.* 1. *&* 2.

$$1.^\circ \quad z^2 + yz + \tfrac{1}{2}y^2 + \tfrac{1}{2}p - \frac{q}{2y} = 0$$

$$2.^\circ \quad z^2 - yz + \tfrac{1}{2}y^2 + \tfrac{1}{2}p + \frac{q}{2y} = 0$$

Subſtituto valore ipſius y, cum valoribus p & q, erit

1.°

$$1^a \quad z^2 - z\sqrt{aa+cc} + \tfrac{3}{4}aa - \frac{a^3 - acc}{2\sqrt{aa+cc}} = 0$$

$$2.^a \quad z^2 + z\sqrt{aa+cc} + \tfrac{3}{4}aa + \frac{a^3 + acc}{2\sqrt{aa+cc}} = 0$$

Multiplicando autem tam numeratorem, quam denominatorem fractionis $\frac{a^3 + acc}{2\sqrt{aa+cc}}$ per $\sqrt{aa+cc}$ remanet idem valor, & prodit $\frac{a^3\sqrt{aa+cc} + acc\sqrt{aa+cc}}{2aa+2cc}$; & dividendo hanc per $aa+cc$, erit $\frac{a\sqrt{aa+cc}}{2}$, ſeu $\frac{1}{2}a\sqrt{aa+cc}$, quæ ponenda eſt in duabus illis ſecundi gradus loco ipſius $\frac{a^3 + acc}{2\sqrt{aa+cc}}$, adeoque fiunt

$$1.^a \quad z^2 - z\sqrt{aa+cc} + \tfrac{3}{4}aa - \tfrac{1}{2}a\sqrt{aa+cc} = 0$$

$$2.^a \quad z^2 + z\sqrt{aa+cc} + \tfrac{3}{4}aa + \tfrac{1}{2}a\sqrt{aa+cc} = 0$$

Ex utraque obtinetur duplex valor ipſius z *per Prop.* 1. *Cap.* 8. ſi diſponantur termini, ut moris eſt, & addatur utrinque quadratum ſemicoefficientis ſecundi termini. Sic ex priori

$$z^2 - z\sqrt{aa+cc} + \tfrac{1}{4}aa + \tfrac{1}{4}cc = -\tfrac{1}{2}aa + \tfrac{1}{2}cc + \tfrac{1}{2}a\sqrt{aa+cc}$$

Extracta radice ſecunda, habetur duplex valor, nempe

$$z = \tfrac{1}{2}\sqrt{aa+cc} \pm \sqrt{-\tfrac{1}{2}aa + \tfrac{1}{4}cc + \tfrac{1}{2}a\sqrt{aa+cc}}$$

Et

Et quia poſitum fuit, $x - \frac{1}{2}a = z$, adeoque $x = z + \frac{1}{2}a$, ſi valori invento ipſius z addatur $+ \frac{1}{2}a$, habebitur valor ejusdem x, qui maxime quærebatur, nimirum

$$x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{1}{4}cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa + cc}}$$

Qui duo valores ipſius x dant longitudinem quæſitam BF; & ſunt iidem, qui nobis a Carteſio (*a*) pauciſſimis verbis indicantur, ſed tyronibus ob multiplicem Aſymmetriam non ita facile occurrunt.

SCHOL. I. *Breviorem aliam præclari hujus problematis ſolutionem, quæ ſecundi gradus æquationem non excedit, trademus inferius, ubi de Geometrica æquationum conſtructione* Cap. 12. *agetur.*

SCHOL. II. *Si loco quadrati detur rectangulum* ABCD, *ex cujus angulo* A *ducenda ſit recta æqualis datæ* M, *ut ſupra, tunc problema eſſet ſolidum. Sit enim* AB = a, BC = b, EF = c, & BF = x, *erit* CF = b — x; *ob triangula* CFE, BFA *ſimilia* $AF = \frac{cx}{b-x}$, & $\overline{AF}^2 = \overline{AB}^2 + \overline{BF}^2$, hoc eſt $\frac{ccxx}{b^2 - 2bx + x^2} = aa + xx$, *unde prodit biquadratica priori ſimilis, ſed quæ nullo modo diviſibilis eſt in duas ſecundi gradus, adeoque in problema ſolidum tranſit. Hinc* Pappus Alexandrinus (*b*) *hoc ipſum problema reſolvit per hyperbolam intra Aſymptotos, & Circulum.*

DEFINITIONES.

I. Omnis æquatio compoſita *Reducibilis* dicitur, quæ per æquationes inferioris gradus exacte dividi, &

(*a*) Geometr. lib. 3. pag. 83. (*b*) Mathemat. Collect. lib. 4. Prop. 31.

& ad inferiorem gradum deprimi poteſt. Sic æquatio quinti gradus, quæ dividi poteſt exacte per duas æquationes inferiores, alteram ſecundi & alteram tertii gradus, dicitur *Reducibilis*.

2. Æquatio compoſita, quæ per nullam æquationem inferioris gradus eſt diviſibilis, dicitur *Irreducibilis*. Ut æquatio ſexti gradus, quæ nec per duas æquationes, unam ſecundi, alteram quarti gradus, nec item per duas tertii gradus dividi exacte poteſt, dicitur *Irreducibilis*.

Schol. *Reductio, de qua hic agimus, eo tendit, ut dignoſcatur, an data æquatio deprimi poſſit ad gradum inferiorem: quod fiet, ſi illa ſit exacte diviſibilis in alias æquationes inferioris gradus. Sic ex. gr. æquatio quinti gradus deprimitur ad unam cubicam & alteram ſecundi gradus, ſi exacte valeat in illas duas dividi. Sine hac doctrina haud ſatis diſtingueretur natura æquationum & problematum, unde ipſæ æquationes oriuntur. Nam problemata inferioris gradus facile confunderentur cum iis, quæ ſunt ſuperioris ordinis, & horum inſtar reſolverentur, quod eſt abſurdum. Utimur autem methodo generali & pulcherrima; digna ſane, cui tyrones ſatis diligenter incumbant.*

PROPOSITIO VI.

An Æquatio biquadratica reducibilis ſit, explorare.

Sit æquatio quarti gradus indeterminata, & generalis omnes alias repræſentans $x^4 + px^3 + qx^2 + rx + s = 0$, quæri-

quæritur an dividi poſſit exacte in duas alias ſecundi gradus, & quænam illæ ſint.

Concipiatur illa orta ex duabus indeterminatis ſecundi gradus $x^2+fx+g=0$, & $x^2+bx+i=0$, quæ ſi invicem multiplicentur, oritur æquatio priori æqualis, nempe

$$\begin{aligned} x^4 + fx^3 &+ gx^2 + gbx + ig = 0 \\ + bx^3 &+ fbx^2 + ifx \\ &+ ix^2 \end{aligned}$$

Comparentur termini utriusque æquationis, & fiant æquationes particulares, erit

$1^a.\ f+b=p$, $2^a.\ g+fb+i=q$, $3^a.\ gb+if=r$, $4.\ ig=s$.

Sumatur ex prima valor b, erit $b=p-f$, & ex quarta valor $i=\frac{s}{g}$, qui duo valores ſubſtituantur in tertia æquatione loco b & i; fiet $pg-fg+\frac{fs}{g}=r$, & multiplicando per g, erit $pgg-fgg+fs=rg$, hoc eſt $fs-fgg=rg-pgg$; unde habetur $f=\frac{rg-pgg}{s-gg}$.

Cujus valor, ut plane innoteſcat, innoteſcere debet g. Eſt autem ex quarta æquatione $ig=s$; proinde g eſt unus ex diviſoribus ultimi termini cogniti s, quo cum ſigno $+$ vel $-$ dividi poteſt exacte ultimus terminus s. Tentandi ſunt igitur ſub utroque ſigno $\pm$ ſinguli diviſores, ut docuimus *Propoſ. 1. Cap. 7.* Sed exemplo res illuſtrabitur.

I. Sit æquatio data $x^4-4x^3-10x^2-28x-8=0$ erit $p=-4$, $q=-10$, $r=-28$, $s=-8$. Ut inve-

inveniatur quantitas g, tentandi ſunt omnes diviſores ultimi termini -8 ſub utroque ſigno, nempe ± 1, ± 2, ± 4, ± 8. Ponatur primo eſſe $g = 1$, erit ex ſuperiori formula $f = \frac{-28+4}{-8-1} = \frac{24}{9} = \frac{8}{3}$. Item $h = -4 - \frac{8}{3} = -\frac{20}{3}$, & $i = -8$. His autem valoribus poſitis in duabus indeterminatis $x^2 + fx + g = 0$, & $x^2 + hx + i = 0$, fiunt $x^2 + \frac{8}{3}x + 1 = 0$, & $x^2 - \frac{20}{3}x - 8 = 0$, per quas ſi dividatur æquatio data $x^4 - 4x^3 - 10x^2 - 28x - 8 = 0$, diviſio exacte non ſuccedit. Non eſt igitur $g = 1$, ut fuit ſuppoſitum.

Neque item diviſio ſuccedit, ſi ponatur $g = -1$, vel $g = +2$, vel $g = -2$, bene tamen ſi ponatur $g = 4$; tunc enim erit $f = 2$, $h = -6$, & $i = -2$. Quibus valoribus in duabus illis æquationibus ſecundi gradus ſubſtitutis, habentur $x^2 + 2x + 4 = 0$, & $x^2 - 6x - 2 = 0$, per quas æquatio data exacte dividitur, ejusque radices ex reſolutione duarum illarum ſecundi gradus facile obtinentur *per Propoſ. 2. Cap. 8.* nempe $x = -1 \pm \sqrt{-3}$, & $x = 3 \pm \sqrt{11}$.

II. Sit æquatio $x^4 - x^3 - 5x^2 + 12x - 6 = 0$, ea ipſa, quam Newtonus in Arithm. Univerſ. etiam per *Diviſores ſurdos* reſolvit. Concipiatur orta ex duabus illis indeterminatis ſecundi gradus $x^2 + fx + g = 0$, & $x^2 + hx + i = 0$. Inventum jam fuit ſupra $f = \frac{rg - pgg}{s - gg}$, $h = p - f$, $i = \frac{s}{g}$. Comparatis autem æquationis datæ terminis cum formula generali, habetur $p = -1$, $q = -5$,

$r = 12$,

$r = 12$, $s = -6$; & ultimi termini (-6) diviſores, ex quibus g (ob $ig = s$) eſt unus, ſunt 1, 2, 3, 6, qui ponantur ſingulatim in ſuperiori formula loco g cum ſigno $\pm$. Et cum fruſtra adhibeantur ± 1, ± 2, pono ipſum $g = 3$; unde $f\left(= \frac{rg - pgg}{s - gg}\right) = \frac{36 + 9}{-6 - 9} = \frac{45}{-15} = -3$: item $h\,(= p - f) = -1 + 3 = 2$; & $i\left(= \frac{s}{g}\right) = \frac{-6}{3} = -2$.
Per hos valores determinatis duabus ſecundi gradus æquationibus, habetur $x^2 - 3x + 3 = 0$, & $x^2 + 2x - 2 = 0$, quæ æquationem datam exacte dividunt, earumque radices $x = 1\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$, & $x = -1 \pm \sqrt{3}$ quatuor æquationis datæ radices exhibent.

COROLL. *Subſtitutis ipſarum* f, g, h, *&* i *valoribus in duabus illis ſecundi gradus æquationibus, ſi æquatio data per neutram poſſit exacte dividi, ſignum eſt, eam eſſe irreducibilem. Tunc autem ejus ſolutio quærenda eſt* per Prop. 2. hujus, *cum ex natura ſua biquadratica ſit: quemadmodum cubicas quæ deprimi nequeunt ad gradum inferiorem, per* Cardani *regulas ſolvimus.*

PROPOSITIO VII.

An Æquatio quinti gradus reducibilis ſit, inquirere.

Sit æquatio generalis repræſentans omnes quinti gradus æquationes $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$, quæritur an exacte dividi poſſit in alias duas inferioris gradus, alteram ſcilicet ſecundi, alteram tertii gradus, & quænam ſint tales æquationes. Supponantur illæ duæ æqua-

 tiones

tiones eſſe $x^2+fx+g=0$, & $x^3+bx^2+ix+l=0$, ex quarum facto habetur æquatio priori ex hypotheſi æqualis, nempe

$$\begin{array}{l} x^5+fx^4+ix^3+lx^2+flx+gl=0 \\ \quad +bx^4+gx^3+gbx^2+gix \\ \qquad +fbx^3+fix^2 \end{array}$$

Comparatis terminis, habentur quinque æquationes: 1ª $f+b=p$, 2ª $i+g+fb=q$, 3ª $l+gb+fi=r$, 4ª $fl+gi=s$, & 5ª $gl=t$.

Ex prima æquatione habetur $b=p-f$, & ex ſecunda $b=\frac{q-i-g}{f}$; erit ergo $p-f=\frac{q-i-g}{f}$, & multiplicando per f, oritur $pf-ff=q-i-g$.

Ex hac æquatione ſumatur valor i, erit $i=ff-pf+q-g$, & ex quarta valor alter ipſius i, nempe $i=\frac{s-fl}{g}$, & in hac ſubſtituto valore l, qui ex æquatione quinta deſumitur (hoc eſt $l=\frac{t}{g}$) erit $i=\frac{s}{g}-\frac{ft}{gg}$, proinde oritur $ff-pf+q-g=\frac{s}{g}-\frac{ft}{gg}$, & ordinando æquationem reſpectu ad incognitam f, habetur æquatio ſecundi gradus

$$\begin{array}{l} ff-pf+q=0 \\ \quad +\frac{tf}{gg}-g \\ \qquad -\frac{s}{g} \end{array}$$

Ex

Ex qua ut obtineatur valor ipſius f, determinari prius debet quantitas g, quæ habetur ut cognita, cum ſit unus ex diviſoribus ultimi termini æquationis datæ: nam ex quinta æquatione $gl = t$. Ponatur ergo $g = 1$. Sed claritatis gratia ecce exemplum.

Sit data æquatio $x^5 - 4x^4 + 6x^3 - 8x^2 + 5x - 1 = 0$, erit $p = -4$, $q = 6$, $r = -8$, $s = 5$, $t = -1$. Poſitis valoribus p, q, s, t in illa æquatione ſecundi gradus cum valore g, fiet illa $ff + 3f = 0$, unde $f = -3$. Ponantur hi valores $f = -3$, & $g = 1$ in æquatione $x^2 + fx + g = 0$, oritur $x^2 - 3x + 1 = 0$, per quam dividi debet æquatio ſuperior data $x^5 - 4x^4 + 6x^3$ &c. quæ diviſio exacte ſuccedit, & habetur in quoto $x^3 - 1x^2 + 2x - 1 = 0$, quæ quidem amplius diviſibilis non eſt. Habentur ergo duæ æquationes quæſitæ, & æquationem datam eſſe reducibilem concluditur. Radices æquationis ſecundi gradus habentur *per Prop. 1. & 3. Cap. 8.* Cubicæ vero *per Prop. 8. vel 9. Cap. 9.*

COROLL. I. *Manifeſtum eſt, æquationem* $x^3 - 1x^2 + 2x - 1 = 0$, *quæ provenit ex diviſione propoſitæ æquationis, habere pro coefficientibus valores indeterminatarum* h, i, l, *quæ ſuppoſitæ fuerunt, nempe* -1, $+2$, -1, *proinde æquationem datam per alterutram potuiſſe dividi. Quod & pro ſequenti Propoſ. intelligitur.*

COROLL. II. *Subſtitutis ſingulis ultimi termini diviſoribus* $+1$, -1 *in æquatione* $x^2 + fx + g = 0$, *ſi per hanc æquatio data exacte dividi minime potuiſſet, ſignum erat, illam eſſe irreducibilem, & ſolidum quinque dimenſionum problema proprie conſtituere, cujus latera per Curvas inveniri debent.*

PRO-

PROPOSITIO VIII.

An æquatio ſexti gradus ſit reducibilis, examinare.

Sit æquatio generalis repræſentans omnes alias ſexti gradus æquationes $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$, quæritur an diviſibilis ſit in alias duas inferioris gradus, & quænam illæ ſint.

I. Supponatur eſſe altera quarti gradus $x^4 + fx^3 + gx^2 + hx + i = 0$, altera ſecundi gradus $x^2 + mx + n = 0$. Ex quarum facto oritur æquatio priori æqualis, nempe

$$\begin{array}{llllll} x^6 + fx^5 & + gx^4 & + hx^3 & + ix^2 & + imx & + in = 0 \\ \quad + mx^5 & + fmx^4 & + gmx^3 & + hmx^2 & + hnx & \\ & + nx^4 & + fnx^3 & + gnx^2 & & \end{array}$$

Unde facta terminorum utriusque æquationis comparatione, habentur ſex æquationes. 1ª $f + m = p$, 2ª $g + fm + n = q$, 3ª $h + gm + fn = r$, 4ª $i + hm + gn = s$, 5ª $im + hn = t$, 6ª $in = v$.

Sumatur ex prima valor f, nempe $f = p - m$, quo in ſecunda æquatione poſito, oritur $g + pm - mm + n = q$, unde eruitur valor $g = q - pm + mm - n$. Cum autem habeatur in ſexta $in = v$, erit $i = \frac{v}{n}$, quo valore poſito in quinta æquatione, habetur $\frac{mv}{n} + hn = t$, ex qua eruitur valor h, nempe $h = \frac{t}{n} - \frac{mv}{nn}$. Jam valor

lor ipſarum g, h, i, ponatur in quarta æquatione, oritur $\frac{v}{n}+\frac{tm}{n}-\frac{m^2v}{nn}+qn-pmn+m^2n-nn=s$. Duæ fractiones $\frac{v}{n}+\frac{tm}{n}$ reducantur ad idem nomen cum alia $\frac{-m^2v}{nn}$, fiet æquatio $\frac{vn+tmn-m^2v}{nn}+qn-pmn+m^2n-nn=s$, & multiplicando omnes terminos per nn, ut fractio evaneſcat, fit $vn+tmn-m^2v+qn^3-pmn^3+m^2n^3-n^4=snn$, hoc eſt

$$m^2n^3-m^2v=pmn^3-tmn-vn-qn^3+n^4+sn^2.$$

Dividendo autem utrumque æquationis membrum per n^3-v, & ordinando æquationem ſecundi gradus in ordine ad incognitam m, habetur

$$\frac{\begin{array}{l} m^2-pn^3m+vn=0 \\ \quad\;\, +\;tmn\;+qn^3 \\ \qquad\qquad\;\;\; -n^4 \\ \qquad\qquad\;\;\; -sn^2 \end{array}}{n^3-v}$$

Eſt autem quantitas n unus ex diviſoribus ultimi termini æquationis, nam ex ſexta æquatione habetur $in=v$. Itaque diviſores ſinguli, quibus æquari poteſt n, cum ſignis + & — ſubſtituendi ſunt ſucceſſive in ſuperiori æquatione una cum valoribus p, q, t, s, v ad obtinendum valorem ipſius m. Sed claritatis gratia,

Sit æquatio data $x^6 - 13x^5 + 45x^4 - 71x^3 + 57x^2 - 16x + 2 = 0$, erit $p = -13$, $q = 45$, $r = -71$, $s = 57$, $t = -16$, $v = 2$. Ponatur $n = 2$. Æquatio ſuperior ſecundi gradus fit $m^2 + 12m + 20 = 0$, unde *per Prop.* 1. *Cap.* 8. habetur valor ipſius $m = -2$, ſubſtituendus in æquatione aſſumpta $x^2 + mx + n = 0$ una cum valore n, quæ proinde fit $x^2 - 2x + 2 = 0$. Per hanc tentari debet diviſio æquationis datæ $x^6 - 13x^5 + 45x^4$ &c. quæ quidem exacte dividitur, & in quoto habetur æquatio altera quarti gradus $x^4 - 11x^3 + 21x^2 - 7x + 1 = 0$, quæ amplius diviſibilis non eſt. Æquatio autem $x^2 - 2x + 2 = 0$ exhibet duas radices $x = 1 \pm \sqrt{-1}$ *per Prop.* 1. *Cap.* 8. reliquæ habentur *per Propoſ.* 2. *hujus*, aut per Curvas ſi biquadratica ſolutionem non admittat.

II. Quod ſi, his peractis, æquatio data ſexti gradus dividi non potuiſſet per æquationem ſecundi gradus, ut ſupra, inventam, tunc videndum, an illa ex duabus tertii gradus fuerit compoſita, ideoque in duas tertii gradus ſit diviſibilis. Sit igitur una ex his $x^3 + fx^2 + gx + h = 0$, altera $x^3 + mx^2 + nx + l = 0$, ex quarum facto oritur

$$\begin{array}{llllll} x^6 + fx^5 & + gx^4 & + hx^3 & + hmx^2 & + hnx & + hl = 0 \\ \quad + mx^5 & + fmx^4 & + gmx^3 & + gnx^2 & + glx & \\ & + nx^4 & + fnx^3 & + flx^2 & & \\ & & + lx^3 & & & \end{array}$$

Quæ ex hypotheſi eſt æqualis æquationi generali repræſentanti omnes ſexti gradus æquationes, ſcilicet

$$x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$$

Facta

Facta igitur terminorum comparatione, habentur ſex æquationes, 1.ª $f + m = p$, 2.ª $g + fm + n = q$, 3.ª $h + gm + fn + l = r$, 4.ª $hm + gn + fl = s$, 5.ª $hn + gl = t$, 6.ª $hl = v$.

Cum ex prima deducatur $f = p - m$, multiplicando per m, erit $fm = pm - m^2$. Ponatur in ſecunda æquatione loco fm ejus valor, fiet $g + pm - m^2 + n = q$, hoc eſt $g = q - pm + m^2 - n$, ſeu $m^2 - pm - n + q = g$: quæ quidem æquatio ſi ducatur in l, erit $m^2 l - pml - nl + ql = gl$; ſed ex quinta $gl = t - hn$, erit ergo $m^2 l - pml - nl + ql = t - hn$. Utraque ducatur in l, oritur $m^2 ll - pmll - nll + qll = tl - hnl$. Cum autem ex ſexta ſit $hl = v$, ſubſtituto hoc valore, habetur $m^2 ll - pmll - nll + qll = tl - vn$, ſeu $m^2 ll - pmll + qll - tl = nll - vn$, unde eruitur valor $n = \frac{m^2 ll - pmll + qll - tl}{ll - v}$.

Inveniri debet jam alter ipſius n valor hac ratione. Ducatur æquatio tertia $h + gm + fn + l = r$ in l, oritur $hl + gml + fnl + ll = rl$; cumque ex ſexta habeatur $hl = v$, ſubſtituto hoc valore, erit $v + gml + fnl + ll = rl$, ſeu $gml = rl - v - fnl - ll$. Habetur autem ex ſecunda æquatione $g = q - fm - n$, quæ ſi multiplicetur per ml, oritur $gml = qml - fm^2 l - mnl$; hinc $rl - v - fnl - ll = qml - fm^2 l - mnl$. Ponatur loco f ejus valor ex prima æquatione deſumptus, erit $rl - v - pnl + mnl - ll = qml - pm^2 l + m^3 l - mnl$, ſeu $2mnl - pnl = m^3 l - pm^2 l + qml + ll - rl + v$, atque hinc tandem obtinebitur $n = \frac{m^3 l - pm^2 l + qml + ll - rl + v}{2ml - pl}$.

Ex duobus ejuſdem x valoribus inventis deducitur

$$\frac{m^2ll - pmll + qll - tl}{ll - v} = \frac{m^3l - pm^2l + qml + lt - rl + v}{2ml - pl}$$

Factaque invicem multiplicatione, & ordinata æquatione tertii gradus, ad quem aſcendit indeterminata m, erit

$$m^3 - 2pl^3m^2 - 2tl^2m - pql^3 = 0$$
$$\begin{array}{r} - vplm^2 + p^2l^3m + ptl^2 \\ + ql^3m - l^4 \\ + vqlm + rl^3 \\ - vrl \\ + vv \\ \hline l^3 + vl \end{array}$$

Quæritur modo valor ipſius m; ſed prius determinari debet l, qui eſt unus ex diviſoribus ultimi termini datæ æquationis. Nam $bl = v$ ex ſexta æquatione. Utque id omne clarius intelligatur,

Sit æquatio $x^6 - 8x^5 + 13x^4 - 23x^3 + 10x^2 - 7x - 12 = 0$. Erit $p = -8$, $q = 13$, $r = -23$, $s = 10$, $t = -7$, $v = -12$, cujus diviſores ſunt 1, 2, 3, 4, 6, 12, *per Prop.* 9. *Cap.* 1. Ponatur igitur $l = 1$, & ſingulis valoribus p, q, r, t, l in ſuperiori æquatione ſubſtitutis, invenitur æquatio ipſa $m^3 + \frac{80m^2 + 65m - 4}{11} = 0$.
Hæc ut a fractione liberetur *per Propoſ.* 10. *Cap.* 6. fiat $m = \frac{y}{11}$, oritur $y^3 + 80y^2 + 715y - 484 = 0$, quæ quidem

dem ſi dividatur per binomium $y + 11$, hoc eſt per unum ex diviſoribus ultimi termini inventum *per Prop.* 1. *vel* 2. *Cap.* 7. diviſio fit ſine reſiduo, proinde innoteſcit $y = -11$, *per Schol.* 1. *Prop.* 1. *Cap.* 6. atque hinc $m = \frac{y}{11} = \frac{-11}{11} = -1$. Hoc valore ipſius m una cum valoribus p, q, t, l poſitis in æquatione $n = \frac{m^2 ll - pmll + ql - tl}{ll - v}$ fit $n = \frac{13}{13} = 1$. Determinatis ergo jam valoribus m, n, l, determinata quoque exiſtit æquatio aſſumpta $x^3 + mx^2 + nx + l = 0$, quæ fit $x^3 - 1x^2 + 1x + 1 = 0$, per quam fieri debet diviſio æquationis datæ $x^6 - 8x^5 + 13x^4 - 23x^3 + 10x^2 - 7x - 12 = 0$, quæ quidem exacte ſuccedit, & habetur in quoto $x^3 - 7x^2 + 5x - 12 = 0$, quæ amplius non eſt diviſibilis. Patet ergo æquationem datam reducibilem eſſe in duas tertii gradus $x^3 - 7x^2 + 5x - 12 = 0$, & $x^3 - 1x^2 + 1x + 1 = 0$. Quarum radices per Cardani regulas *ex Propoſ.* 8. *Cap.* 9. obtineri facile poſſunt.

Coroll. *Si æquationis diviſio haud ſucceſſiſſet, ſubſtituendi erant ſucceſſive loco* l *ſinguli diviſores ipſius* $v = -12$. *At ſi diviſoribus his fruſtra tentatis, diviſio fieri non poſſit, æquatio erit revera ſexti gradus & irreducibilis.*

Schol. *Plerique de his reductionibus agunt, cum æquationum compoſitarum radices rationales inquirunt, ad quas proprie pertinent. At differre huc nobis placuit, quod tyrones ſine prævia reſolutione æquationum eas haud ſatis percipere valeant, ut ex reductionibus æquationum quinti & ſexti gradus ſupra allatis evidens eſt.*

CAPUT XI.

De limitibus radicum, earumque approximatione.

PROPOSITIO I.

Radicum limites invenire.

INveniri debent per Analyſim duæ quantitates, inter quas radix æquationis continetur, quæ limites dicuntur, hoc pacto:

I. Sit æquatio $x^2 + px - q = 0$, erit $x^2 + px = q$, & ſublato ex uno membro x^2, remanet $px < q$, atque hinc $x < q : p$ Duo puncta (:) ſunt nota diviſionis.

Similiter quia $x^2 + px = q$, ſublato ex una parte px, erit $q > x^2$, & $\sqrt{q} > x$. Multiplicando autem per x, erit $x\sqrt{q} > x^2$, & addendo utrinque px, habetur $x\sqrt{q} + px > x^2 + px$; ſed $x^2 + px = q$, proinde erit quoque $x\sqrt{q} + px > q$; atque hinc habetur $x > \frac{q}{p + \sqrt{q}}$. Sunt ergo limites quæſiti $q : p$, & $q : (p + \sqrt{q})$ unde propoſitæ æquationis radix minor eſt $q : p$, & major $q : p + \sqrt{q}$.

Sit exemplum $x^2 + 4x - 12 = 0$, erit $p = 4$, $q = 12$, unde $q ; p = 12 : 4 = 3$; & $q : (p + \sqrt{q}) = 12 : 4 + 3 = 12 : 7 = 1\frac{5}{7}$. Sunt ergo limites quæſiti 3, & $1\frac{5}{7}$ proinde radix nequit eſſe major quam 3, nec minor quam $1\frac{5}{7}$. Equidem ſi tentetur diviſio per $x + 2$, minime ſuccedit, bene

bene tamen ſuccedit exacte per $x-2$, & habetur in quoto altera æquationis radix $x+6$. Unde vel hoc uno exemplo patet, beneficio limitum nos multo labore liberari, cum ex omnibus ultimi termini diviſoribus duo tantum tentandi fuerint $x+2$, & $x-2$. *Vide Coroll. 2. hujus.*

II. Sit $x^2-px+q=0$, erit $x^2+q=px$, proinde $x^2<px$, & dividendo per x, $x<p$. Similiter quia $x^2+q=px$, erit $q<px$; & $q:p<x$. Sunt ergo limites quæſiti p, & $q:p$, ita ut radix minor ſit quantitate cognita p, ſed major quam $q:p$.

Sit exemplum $x^2-3x+3=0$, erit $p=3$, & $q=3$, cujus limites ſunt $p=3$, & $q:p=3:3=1$. Radix ergo non erit major quam 3, nec minor unitate, nempe $x=1\frac{1}{2}\pm\sqrt{-\frac{3}{4}}$. *Vide Coroll. 3. hujus.*

III. Sit $x^2-px-q=0$, erit $x^2=px+q$, proinde $x^2>q$, ideoque $x>\sqrt{q}$, & multiplicando utrinque per $\sqrt{q}$, habetur $x\sqrt{q}>q$. Jam ſi in æquatione $x^2=px+q$ loco ipſius q ponatur $x\sqrt{q}$, erit $x^2<px+x\sqrt{q}$, & dividendo per x, erit $x<p+\sqrt{q}$.

Pariter quia $x^2=px+q$, erit $x^2>px$, & dividendo per x, erit $x>p$, & multiplicando utrinque per p, erit $px>pp$; ſed $x^2=px+q$, ergo $x^2>pp+q$, & extrahendo utrinque radicem ſecundam, habetur $x>\sqrt{pp+q}$. Sunt ergo limites quæſiti $p+\sqrt{q}$, & $\sqrt{p^2+q}$, hoc eſt radix debet eſſe minor quam $p+\sqrt{q}$, ſed major quam $\sqrt{pp+q}$.

IV. Sit æquatio cubica ſecundo termino carens $x^3-qx+r=0$, erit $x^3=qx-r$, unde $x^3<qx$, & dividendo

do

do per x, $x^2 < q$; ideoque $x < \sqrt{q}$. Similiter cum ſit ex hypotheſi $x^3 - qx + r = 0$, erit $x^3 + r = qx$, atque hinc $qx > r$, & $x > r:q$. Sunt ergo limites quæſiti $\sqrt{q}$, & $r:q$, ita ut radix æquationis nequeat eſſe major $\sqrt{q}$, nec minor $r:q$.

Sit exemplum $x^3 - 19x + 30 = 0$, erit $q = 19$, $r = 30$, unde $\sqrt{q} = 4$, & $r:q = \frac{30}{19} = 1\frac{1}{2}$ circiter. Tentanda eſt ergo diviſio per $x + 2$, vel $x - 2$, vel $x + 3$, aut $x - 3$, & quidem exacte ſuccedit per $x - 2$, & habetur in quoto æquatio ſecundi gradus $x^2 + 2x - 15 = 0$, cujus una radix eſt $+3$, altera -5 *per Prop.* 1. *vel* 3. *Cap.* 8.

V. Sit æquatio cubica $x^3 - px - q = 0$, erit $x^3 - px = q$, & dividendo per x primam partem, fit $x^2 - p < q$, adeoque $x^2 < p + q$, & $x < \sqrt{p+q}$. Similiter cum ſit $x^3 - px = q$, auferendo ex prima parte px, erit $x^3 > q$, adeoque $x > \sqrt[3]{q}$.

Sit exemplum $x^3 - 12x - 12 = 0$, erit $p = 12$, & $q = 12$; adeoque $x < \sqrt{24} = 4$, item $x > \sqrt[3]{12} = 2$. Eſt ergo x minor quam 4, ſed major quam 2, proinde radix inter 4 & 3 latet, ut inferius patebit.

VI. Demum ſit æquatio cubica $x^3 + qx - r = 0$, erit $x^3 + qx = r$; & ſubtrahendo ex una tantum parte x^3 erit $qx < r$; dividendo autem per q, habetur $x < r:q$ pro primo limite.

Similiter quia $x^3 + qx = r$, ſublato ex una tantum parte qx, erit $x^3 < r$, & $x < \sqrt[3]{r}$, & elevando ad ſecundam

poten-

potentiam utrunque membrum, erit $xx < \sqrt[3]{r^2}$, & multiplicando per x, erit $x^3 < x\sqrt[3]{r^2}$. Hoc autem valore in æquatione data ſubſtituto, habetur $qx + x\sqrt[3]{r^2} > r$, proinde $x > \frac{r}{q+\sqrt[3]{r^2}}$ dat alterum limitem quæſitum. Radix igitur minor eſt $r : q$, ſed major $\frac{r}{q+\sqrt[3]{r^2}}$.

VI. Sit æquatio quarti gradus $x^4 - px^2 - qx - r = 0$, erit $x^4 - px^2 = qx + r$, proinde $x^4 > px^2$, alias in hac ipſa æquatione $qx + r$ non eſſet quantitas poſitiva. Eſt ergo $x^2 > p$, & $x > \sqrt{p}$.

Similiter quia $x^4 - px^2 - qx = r$, ſi dividatur primum membrum per x, erit $x^3 - px - q < r$, pariterque $x^3 - px < r + q$. Dividatur primum membrum per x, erit multo magis $x^2 - p < r + q$, item $x^2 < r + q + p$, & extracta radice, habetur $x < \sqrt{r+q+p}$. Sunt ergo limites $\sqrt{p}$, & $\sqrt{r+q+p}$.

Sit exemplum $x^4 - 8x^2 - 4x + 3 = 0$, erit $p = 8$, $q = 4$, $r = 3$; hinc $\sqrt{p} = 2$, & $\sqrt[4]{r+q+p} = 1+2+2 = 5$. Eſt ergo radix inter limites 2 & 5.

COROLL. I. *Eadem ratione inveniri poſſunt limites pro iiſdem æquationibus etiamſi nullo termino careant; tum etiam pro æquationibus altiorum graduum, quemadmodum egregie fecit* Florimundus de Beaune (*a*), *cui tam præclarum inventum, teſte* Eraſmo Bartolino, *debemus acceptum.*

(*a*) Introduct. ad Geometriam Carteſii Amſtelodami an. 1683. pag. 121.

Coroll. II. *In limitibus dignoſcendis numeri integri ſufficiunt; ideoque fractiones tuto negligi poſſunt, præſertim in extractione radicum. Item ex radicalibus, quæ in formulis limitum apparent, ſatis eſt radicem extrahere proxime minorem, neglectis fractionibus, ut limes ſit quantitas a fractionibus immunis & rationalis.*

Coroll. III. *Quanquam in æquatione generali quantitates* p, q, r *&c. denotent quantitates negativas, in ſpeciali tamen æquatione quantitates omnes tanquam poſitivæ accipiuntur, ut exempla ſuperiora ſatis docent.*

PROPOSITIO. II.

Radicum limites pro quacunque æquatione methodo Newtoniana indagare.

I. Multiplicentur ſinguli datæ æquationis termini per numerum dimenſionum, quas in illis habet incognita, hoc eſt per progreſſionem Arithmeticam naturalem deſcendentem, ita ut ultimus terminus ſit zero, & factum dividatur per radicem æquationis. Deinde æquatio, quæ oritur uno gradu inferior, multiplicetur per aliam ſimilem progreſſionem Arithmeticam, & hoc factum ſimiliter dividatur per radicem æquationis; quod quidem toties fiet, donec occurrat æquatio ſimplex linearis.

Sit æquatio	*A*	$x^3 - x^2$	$- 14x$	$+ 24$	$= 0$
Progr.		3. 2.	1.	0.	
Div. per x		$3x^3 - 2x^2$	$- 14x$	$= 0$	

B $3x^2$

$$\begin{array}{lll} & B & 3x^2 - 2x - 14 = 0 \\ \textit{Pro'r.} & & 2. \quad 1. \quad 0. \\ \hline & & 6x^2 - 2x = 0 \\ \textit{Div. per } x & & 6x - 2 = 0 \\ & C & 3x = 1 \end{array}$$

Æquationes *A*, *B*, *C* dicuntur æquationes limitum. Jam vero ad indagandum limitem, intra quem continentur radices poſitivæ datæ æquationis, hæc eſt regula: in æquationibus *A*, *B*, & *C* loco incognitæ x ponitur quilibet numerus 1, vel 2, vel 3 &c. incipiendo ab unitate: nam numerus, qui in ſingulis æquationibus ſummam poſitivam efficit, dat limitem quæſitum. Sic in ſuperiori exemplo ponendo 1 loco x in æquatione *C*, $3x - 1 = 2$, prodit ſumma poſitiva, ſed in æquatione *B*, $3x^2 - 2x - 14 = -13$ prodit negativa. Idem ſequitur ponendo 2, & 3 loco x. Quare infertur limitem his numeris fore majorem. Pono igitur 4, & prodeunt numeri omnes poſitivi ſcilicet

$$3x - 1 = 11$$
$$3x^2 - 2x - 14 = 26$$
$$x^3 - x^2 - 14x + 24 = 16$$

Omnes hi numeri poſitivi indicant numerum 4 eſſe limitem, quem nulla poteſt radicum poſitivarum tranſcendere. Et quidem dividendo ſuperiorem æquationem per $x - 3$, diviſio exacte & ſine reſiduò ſuccedit, adeoque habetur 3 una ex radicibus poſitivis ejuſdem æquationis.

II. Pro inveniendo limite, quem maxima radicum negativarum non tranſcendit, examinandum eſt ſimili ratione, quinam numerus negativus poſitus loco x in iiſdem tribus æquationibus A, B, & C efficiat ſummam poſitivam in æquationibus illis, quæ habent dimenſiones numero pares, ſummam vero negativam in æquationibus dimenſionum numero imparium. In ſuperiori exemplo fruſtra tentantur $-1, -2, -3, -4$: ſed poſito -5 loco x, prodit ſumma negativa in æquationibus numero imparium dimenſionum, poſitiva vero in æquationibus dimenſionum numero parium; proinde -5 eſt limes, quem radix negativa non tranſcendit, videlicet

$$3x - 1 = -16$$
$$3x^2 - 2x - 14 = +71$$
$$x^3 - x^2 - 14x + 24 = -56$$

Itaque ſi ſuperior æquatio data $x^3 - x^2 - 14x + 24 = 0$ dividatur per $x + 4$, diviſio exacte ſuccedit, unde habetur radix negativa -4.

III. Pro obtinendo radicum negativarum limite adhiberi poſſunt etiam numeri affirmativi, modo mutentur ſigna terminorum parium in æquationibus limitum, quod in ſequenti exemplo factum videbis, tunc enim radices falſæ fiunt veræ *per Prop.* 3. *Cap.* 6. ideoque aſſumi poſſunt numeri affirmativi.

Sit æquatio $x^5 - 1x^4 - 7x^3 + 1x^2 + 4x - 4 = 0$, facta eadem operatione, ut ſupra, habentur æquationes limitum, ſcilicet

$$5x - 1 = 0$$
$$10x^2 - 4x - 7 = 0$$
$$10x^3 - 6x^2 - 21x + 1 = 0$$
$$5x^4 - 4x^3 - 21x^2 + 2x + 4 = 0$$
$$x^5 - 1x^4 - 7x^3 + 1x^2 + 4x - 4 = 0$$

In quibus ſi loco x ſubſtituantur 1, vel 2, vel 3, non ubique prodit ſumma poſitiva; ſubſtituendo autem 4, ſumma oritur in ſingulis poſitiva, proinde 4 eſt quantitas, quam radicum poſitivarum maxima non tranſgreditur.

Ad inveniendum limitem, quem radicum negativarum maxima non excedit, mutatis terminorum parium ſignis, adhibentur eodem modo numeri affirmativi. Itaque ponendo in ſingulis æquationibus loco x numerum 2, prodit ubique ſumma affirmativa. In æquatione tamen $x^5 + 1x^4 - 7x^3 + 1x^2 + 4x + 4 = 0$, prodit zero, quod indicat numerum 2 non modo eſſe limitem quæſitum, ſed etiam unam ex radicibus ejuſdem æquationis, quæ proinde facit, ut omnia producta ſe mutuo deſtruant *per num.* 5. *Propoſ.* 1. *Cap.* 6. En limitum æquationes cum ſignis mutatis.

$$5x + 1 = 11$$
$$10x^2 + 4x - 7 = 41$$
$$10x^3 + 6x^2 - 21x - 1 = 61$$
$$5x^4 + 4x^3 - 21x^2 - 2x + 4 = 28$$
$$x^5 + 1x^4 - 7x^3 - 1x^2 + 4x + 4 = 0$$

Habentur ergo limites 4, & −2, inter quos radices omnes æquationis datæ continentur.

Ratio

Ratio methodi hujus eſt, quia in æquatione radicem poſitivam habente omnis numerus poſitivus loco incognitæ x ſubſtitutus vel eſt æqualis ipſi radici, & ſic æquatio illa tota evaneſcit *per num.* 5. *Prop.* 1. *Cap.* 6., vel eſt major, & & tunc efficit omnia illa producta poſitiva eſſe; vel eſt ipſa radice minor, & in hoc caſu aliquod ex illis productis negativum apparet. Sit brevitatis gratia æquatio ſecundi gradus A, $x + 3x - 10 = 0$, cujus radices ſunt 2 & -5; ex ſuperiori canone limitum æquatio erit B $2x + 3 = 0$. Pono in utraque æquatione A & B loco x numerum poſitivum 3 radice ipſa poſitiva 2 majorem; erit A, $9 + 9 - 10 = 8$, & B, $6 + 3 = 9$, ideoque 3 eſt limes æquationis quæſitus. Quod ſi ponatur 1 eadem radice poſitiva 2 minor, erit A, $1 + 3 - 10 = -6$, & B, $2 + 3 = 5$: quod regulæ repugnat, ut patet, cum productum unum ſit negativum. Proinde 1. nequit eſſe datæ æquationis limes.

Similiter in eadem æquatione radix negativa eſt -5. Pono in utraque æquatione A & B loco x numerum negativum -6 radice ipſa negativa majorem; quo quidem ſubſtituto, erit A, $36 - 18 - 10 = 8$; at B, $-12 + 3 = -9$, quod cum regula optime convenit. At ſi ponatur -4, numerus ſcilicet radice ipſa negativa minor, tunc facta ſubſtitutione in iiſdem æquationibus A & B, prodit A, $16 - 12 - 10 = -6$, at B, $-8 + 3 = -5$ quod regulæ adverſatur; cum æquatio A, quæ paris dimenſionis eſt, producta poſitiva debeat efficere. Igitur ex hujuſmodi productis affirmativis & negativis limites æquationum recte inferuntur.

Co-

COROLL. *Inventis limitibus, non ſolum minimo labore inveniuntur radices rationales, ſi quæ ſunt, quas alioquin per ultimi termini diviſores inquirere valde laborioſum eſt; ſed etiam in cognitionem devenimus, an æquatio propoſita radices ſurdas habeat. Nam ſi illa per nullum numerum intra limites conſtitutum exacte dividi poſſit, ſignum eſt, ipſam non alias niſi ſurdas radices habere. Sic æquationis* $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$, *quam Cl. auctor affert in Arithm. Univerſ. radices conſiſtunt inter limites* 2 *&* − 3; *unde tentandi ſunt ſolummodo diviſores* 1, − 1 *&* − 2, *qui inter eos limites continentur, ut dignoſcatur an talis æquatio per eos dividi exacte poſſit, adeoque radices rationales habeat. At quantus labor, ſi ad hanc rem ſub utroque ſigno* ± *omnes ultimi termini* 120 *diviſores tentandi ſint?*

PROPOSITIO III.

Æquationum radices prope veras per approximationem extrahere.

I. SIt æquatio data $x^3 - 12x - 12 = 0$. Inveniantur limites *per Prop.* 1. *vel* 2. *hujus*, qui ſunt 4 & 3; proinde radix, cum ſit media inter 4 & 3, eſt incommenſurabilis, quæ tamen inveniri poterit prope vera hac methodo.

Sumatur ſemidifferentia inter duos limites 4 & 3, nempe $\frac{1}{2}$, quæ addatur limiti minori, ita ut fiat $3\frac{1}{2} = \frac{7}{2}$, & dividatur æquatio data per $x - \frac{7}{2}$, reſiduum eſt cum ſigno

gno —, hoc eſt — 11 $\frac{1}{8}$. Proinde ſumatur iterum ſemidifferentia inter limitem majorem 4 & $\frac{7}{2}$, quæ eſt $\frac{1}{4}$, & addatur ad $\frac{7}{2}$, fiet $= \frac{15}{4}$. Dividatur æquatio data per $x - \frac{15}{4}$, reſiduum adhuc eſt negativum, nempe — 4 $\frac{17}{64}$. Itaque rurſus ſumatur ſemidifferentia inter limitem majorem 4 & $\frac{15}{4}$, nempe $\frac{1}{8}$, quæ addatur ad $\frac{15}{4}$, fit $= \frac{31}{8}$. Dividatur per $x - \frac{31}{8}$ æquatio data, reſiduum adhuc eſt negativum, nempe — $\frac{151}{512}$. Igitur ſemidifferentia inter majorem limitem 4 & $\frac{31}{8}$, quæ eſt $\frac{1}{16}$, addenda eſt ad $\frac{31}{8}$, ut fiat $= \frac{63}{16}$, factaque diviſione æquationis datæ per $x - \frac{63}{16}$, habetur reſiduum poſitivum + 1 $\frac{3263}{4096}$, proinde $\frac{63}{16}$ eſt limes, qui radicem jam tranſcendit, adeoque propinquiores quæſitæ radicis limites ſunt $\frac{63}{16}$ & $\frac{31}{8}$, hoc eſt $\frac{63}{16}$ & $\frac{62}{16}$; & radix ipſa media eſt inter 3 $\frac{15}{16}$ & $\frac{14}{16}$; ideoque differentia non eſt niſi $\frac{1}{16}$.

II. Si radix adhuc propinquior deſideretur, continuari poteſt eodem modo approximatio. Nam in ſuperiori exemplo cum ex additione ſemidifferentiæ $\frac{1}{16}$ facta ad $\frac{31}{8}$ innoteſcat provenire reſiduum poſitivum + 1 $\frac{3263}{4096}$, addi poteſt ad $\frac{31}{8}$ dimidium tantum ejuſdem ſemidifferentiæ $\frac{1}{16}$, nempe $\frac{1}{32}$, ut fiat $= \frac{125}{32}$, factaque diviſione æquationis datæ per $x - \frac{125}{32}$, provenit reſiduum pariter poſitivum + $\frac{25402}{32768}$. Habentur igitur duo limites quæſitæ radicis adhuc propinquiores, nempe $\frac{31}{8}$ & $\frac{125}{32}$, hoc eſt $\frac{124}{32}$ & $\frac{125}{32}$; proinde radix eſt inter 3 $\frac{28}{32}$ & $\frac{29}{32}$, ita ut differentia tantum ſit $\frac{1}{32}$. Hac ratione continuari poteſt approximatio quantum quis voluerit.

COROLL. *Radix ergo datæ æquationis prope vera in notis decimalibus erit inter* 3 · 87500 *&* 3 · 90625, *& differen-*

ferentia $\frac{3125}{100000}$, *quod quidem obtinetur addendo ad numeratores fractionum quinque zeros & dividendo per* 32.

SCHOL. *Pro æquationibus quadraticis radicis approximatio habetur ex communi Arithmetica addendo ad reſiduum tot cyfrarum paria quot libuerit; hinc* $\sqrt{13} = 3.605\overset{\text{IV}}{5}$, *quater additis* 00. Orontius Finæus *id primus docuit.*

PROPOSITIO IV.

Aliæ approximandi rationes expediuntur.

Sit eadem æquatio $x^3 - 12x - 12 = 0$, cujus una radix ſit minor 4, ſed major 3, ut in præcedenti propoſitione dictum eſt. Ponatur eſſe $x = 3 + y$, & hoc valore in æquatione poſito, oritur

$$\begin{aligned} x^3 &= 27 + 27y + 9y^2 + y^3 \\ -12x &= -36 - 12y \\ -12 &= -12 \\ \hline & -21 + 15y + 9y^2 + y^3 \end{aligned}$$

Quia vero y hoc loco denotat fractionem, quæ additur ad 3, ut fiat ad veram radicem approximatio, & fractionum potentiæ continuo decreſcunt, hinc tuto negliguntur $+ 9y^2 + y^3$, quod etiam in ſequentibus obſervabitur. Eſt igitur $15y = 21$, hoc eſt $y = \frac{21}{15} = \frac{7}{5} = 1.\overset{\text{I}}{4}$ (reducendo $\frac{7}{5}$ ad fractiones decimales facilioris calculi gratia) eſt ergo $y = 1.\overset{\text{I}}{4}$. Ponatur jam $x = 3 + 1.\overset{\text{I}}{4} + y$,

hoc eſt $x = 4 . \overset{\text{I}}{4} + y$, & hoc valore in æquatione ſubſtituto, erit

$$\begin{array}{rcl} x^3 & = & 85 . 184 + 58 . 08y \ldots . \\ -12x & = & -52 . 800 - 12 . 00y \ldots \\ -12 & = & -12 . \\ \hline & & +20 . 384 + 46 . 08y \ldots . \end{array}$$

Hinc habetur $46 . \overset{\text{II}}{08}y = -20 . \overset{\text{III}}{384}$, hoc eſt $y = -\frac{20 . \overset{\text{III}}{384}}{46 . \overset{\text{II}}{08}} = -\overset{\text{I}}{4}$. Sed ſi hujus reſiduo diviſionis addantur duæ, aut tres cyfræ ob majorem approximationem, erit quotus $-\overset{\text{IV}}{4423}$, ſubtrahendus (ob ſignum —) ex $x = 4 . \overset{\text{I}}{4}$. hoc eſt ex $4 . \overset{\text{IV}}{4000}$, eritque reſiduum $3 . \overset{\text{IV}}{9577}$ radix æquationis magis magisque approximans.

Quod ſi rurſus ponatur $x = 3 . 9577 + y$, & pari ratione operatio eadem continuetur, multo major fiet ad veram radicem approximatio, ſed hoc ad methodi intelligentiam ſatis.

Coroll. I. *Hinc facile eruuntur approximandi formulæ, quæ ad hanc rem afferri ſolent. Sit enim, ut ſupra, æquatio* $-21 + 15y = 0$. *Si fiat* $p = -21$, *&* $q = 15$, *erit* $qy = -p$, *proinde* $y = -p : q$, *hoc eſt* $y = \frac{21}{15} = 1 . \overset{\text{I}}{4}$, *omnino ut ſuperius.*

Coroll. II. *Vel cum* Hallejo (*a*) *ſit eadem æquatio* — 21

(*a*) In Tranſact. Anglican. n. 210.

$-21 + 15y + 9y^2 = 0$, *& fiat* $p = -21$, $q = 15$, *&* $r = 9$, *erit* $ry^2 + qy = -p$, *& dividendo per* $q + ry$, *habetur* $y = -p : (q + ry)$. *Sed* ex Coroll. 1. *eſt* $y = -p : q$, *ergo* $y = -p : (q - pr : q)$ *proinde* $y = 21 : (15 + \frac{189}{15}) = 21 : (15 + \frac{63}{5}) = 21 : (\frac{138}{5})$, *hoc eſt* $y = \frac{105}{138}$, *& dividendo per* 3, *erit* $y = \frac{35}{46}$: *quæ fractio ſi ad decimales reducatur* per Propoſ. 4. Append. *erit* $y = 76086$. *Hic valor ſi addatur ad* 3 *limitem æquationis, ita ut fiat* $x = 3 + 76086$, *& continuetur eodem modo operatio, invenitur ex eadem* Halleji *formula radix prope vera, vel quæ a vera minimum diſſentiet.*

SCHOL. *Approximandi regulæ hactenus explicatæ ſunt tantum pro æquationibus numericis.*

PROPOSITIO V.

Radices prope veras in æquationibus literalibus indagare.

Sit data æquatio ſecundi gradus $x^2 - 2nx - r^2 = 0$, cujus radix *per Propoſ. 1. vel 3. Cap. 8.* erit $x = n + \sqrt{r^2 + n^2}$, ſeu $x = n + \overline{r^2 + n^2}^{\frac{1}{2}}$ *per Defin. 5. Cap. 3.* Quæritur valor approximatus potentiæ imperfectæ $\overline{r^2 + n^2}^{\frac{1}{2}}$, quod quidem duplici methodo obtineri poteſt.

I. Ex $r^2 + n^2$ extrahatur radix ſecunda, habebitur *per Prop. 11. Cap. 3.* valor quæſitus per ſeriem infinitam ter-

minorum, nempe $r + \frac{n^2}{2r} - \frac{n^4}{8r^3} + \frac{n^6}{16r^5}$ &c. $= \overline{r^2 + n^2}^{\frac{1}{2}}$ $= \sqrt{r^2 + n^2}$, & ſi fiat $n = 1$, erit $r + \frac{1}{2r} - \frac{1}{8r^3} + \frac{1}{16r^5}$ &c.

II. Supponatur $z = \overline{r^2 + n^2}^{\frac{1}{2}}$, erit $zz = r^2 + n^2$, & $zz - r^2 - n^2 = 0$.

Ponatur deinde z æqualis ſeriei infinitæ terminorum, quorum exponentes progreſſionem Arithmeticam, ipſi vero Geometricam ſervent, & multiplicati exiſtant per literas a, b, c, d &c. quæ hoc loco tanquam indeterminatæ conſiderantur. Hi autem termini proportionales conflantur ex quantitate aliqua cognita datæ æquationis, ut n in hoc exemplo, nempe

Sit $z = an^0 + bn^2 + cn^4 + dn^6 +$ &c. (primus terminus $an^0 = a$, nam $n^0 = 1$, quod ſemper advertatur) elevetur ad ſecundam potentiam utrumque hujus æquationis membrum, & ponatur in ſuperiori æquatione $zz - r^2 - n^2 = 0$ loco zz ejus valor, erit

$$\begin{array}{r|l} zz = & a^2 + 2abn^2 + bbn^4 + 2adn^6 \text{ \&c.} \\ & \qquad\qquad\quad + 2acn^4 + 2bcn^6 \text{ \&c.} \\ -n^2 & \qquad\quad - n^2 \\ -r^2 & -r^2 \end{array}$$

Jam ſinguli termini æquentur zero, ut habeantur totidem æquationes particulares ad determinandum valorem ipſarum a, b, c, d &c.

Erit

Erit 1ª. $a^2 = r^2$, atque hinc $a = r$.

2ª. $2ab = 1$, & poſito r loco a, erit $2rb = 1$, ſeu $b = \frac{1}{2r}$.

3ª. $2ac = -bb$, & ſubſtitutis valoribus b & a, erit $2rc = -\frac{1}{4rr} = c = -\frac{1}{8r^3}$.

4ª. $2ad = -2bc$, hoc eſt $ad = -bc$, & poſitis valoribus a, b, c, habetur $d = \frac{1}{16r^5}$ &c.

Surrogentur hi valores in ſerie aſſumpta $an^0 + bn^2 + cn^4 + dn^6$ &c. erit $r + \frac{1}{2r}n^2 - \frac{1}{8r^3}n^4 + \frac{1}{16r^5}n^6$ &c.

Quod ſi ponatur $n = 1$, erit valor quæſitus ipſius $\overline{r^2 + n^2}^{\frac{1}{2}} = r + \frac{1}{2r} - \frac{1}{8r^3} + \frac{1}{16r^5}$ &c. ut ſupra.

Eſt ergo radix propoſitæ æquationis $x = 1 + r + \frac{1}{2r} - \frac{1}{8r^3} + \frac{1}{16r^5}$ &c.

SCHOL. *Poterat ſeries efformari ex altera quantitate cognita* r, *ita ut fieret* $z = ar^0 + br^2 + cr^4$ &c. *ſi* r *minor fuiſſet quam* n. *Nam ſeries, ut ad verum valorem magis accedat, debet eſſe* convergens. *Quod ut fiat, neceſſe eſt, numerator fractionis denominatore ſit minor* per Coroll. 2. Prop. 8. Cap. 2. *unde in hoc caſu quantitas* n *fieret in ſeriei terminis denominator. Hoc ſemper erit in ejusmodi ſeriebus efformandis obſervandum.*

PROPOSITIO VI.

Idem problema alio modo reſolvitur.

I. Sit eadem $x^2 - 2nx - r^2 = 0$, cujus radix per approximationem quæritur. Supponatur ſtatim valor incognitæ x æqualis ſeriei indeterminatæ quorumcunque terminorum, nempe

Sit $x = an^0 + bn^1 + cn^2 + dn^3 + en^4$ &c. Quantitates a, b, c &c. ſunt indeterminatæ, & $n^0 = 1$, unde $an^0 = a$. Elevetur ſeries ad ſingulos gradus, quos habet x in data æquatione, ita ut æquatio transformetur in aliam indeterminatam, ſcilicet

$$\begin{array}{r|rrrrrl} x^2 & = a^2 & + 2abn & + 2acn^2 & + 2adn^3 & + 2aen^4 & \&c. \\ & & & + bbn^2 & + 2bcn^3 & + 2bdn^4 & \&c. \\ -2nx & & -2an & -2bn^2 & -2cn^3 & + ccn^4 & \&c. \\ -r^2 & -r^2 & & & & -2dn^4 & \&c. \end{array}$$

Jam ſinguli termini æquentur zero, ut determinetur valor ipſarum a, b, c, d &c. ex æquationibus particularibus, quæ ſequuntur.

1ª. $a^2 - r^2 = 0$, atque hinc $a^2 = r^2$, & $a = r$.

2ª. $2ab - 2a = 0$, & poſito r loco a, erit $rb = r$, hoc eſt $b = 1$.

3ª. $2ac + bb - 2b = 0$, & $2ac + bb = 2b$, in quâ poſitis valoribus a & b, habetur $2rc + 1 = 2$, atque hinc $c = \frac{1}{2r}$.

4ª. 2ad

4ª. $2ad + 2bc - 2c = 0$, & $ad + bc = c$, in eaque ſubſtitutis valoribus ipſarum a, b, c, erit $d = 0$.

5ª. $2ae + 2bd + cc - 2d = 0$, & abjectis (ob $d = 0$) terminis $2bd - 2d$, remanet $2ae = -cc$, unde eruitur $e = -\frac{1}{8r^3}$.

His valoribus ſurrogatis in ſerie $a + bn^1 + cn^2 + dn^3$ &c. oritur ſeries determinata $r + n^1 + \frac{1}{2r}n^2 - \frac{1}{8r^3}n^4 - \frac{1}{32r^5}n^6$ &c. ac poſito $n = 1$, erit $x = r + 1 + \frac{1}{2r} - \frac{1}{8r^3} - \frac{1}{32r^5}$ &c. Ex his pauciſſimis terminis patet, hanc ſeriem magis, quam duas ſuperiores, eſſe convergentem.

II. Sit æquatio cubica
$$\begin{array}{l} x^3 + bbx - 2b^3 = 0 \\ \qquad + abx - a^3 \end{array}$$

Fiat ſeries ex quantitate cognita omnium minima, quæ ſit a, multiplicata per indeterminatas f, g, h, i &c. & eidem æqualis ponatur incognita x, erit $x = fa^0 + ga^1 + ha^2 + ia^3 + la^4$ &c. quæ ſeries elevetur ad omnes gradus, quos habet x in æquatione data, erit

$$\begin{array}{r|llllll}
x^3 = & f^3 & + 3ffga & + 3fgga^2 & + g^3a^3 & + 3fhha^4 & \&c. \\
 & & & + 3ffha^2 & + 3ffia^3 & + 3ggha^4 & \&c. \\
 & & & & + 6fgha^3 & + 3ffla^4 & \&c. \\
 & & & & & + 6fgia^4 & \&c. \\
+ bbx & + bbf & + bbga & + bbha^2 & + bbia^3 & + bbla^4 & \&c. \\
+ abx & & + bfa & + bga^2 & + bha^3 & + bia^4 & \&c. \\
- 2b^3 & - 2b^3 & & & & & \\
- a^3 & & & & - a^3 & &
\end{array}$$

Fiant

Fiant ſinguli termini zero æquales, ut habeantur totidem æquationes particulares ad determinandum valorem ipſarum f, g, h, i &c.

1ª $f^3+fhh-2h^3=0$, quæ ſi dividatur per $f-h$, diviſio exacte ſuccedit, proinde h eſt radix vera ejusdem æquationis, *per num. 5. Propoſ. 1. Cap. 6.* eritque $f=h$.

2ª. $3ffg+hhg+hf=0$, & $3ffg+hhg=-hf$, & poſito h loco f, habetur $g=-\frac{1}{4}$. Nam $g=-\frac{hh}{4hh}=-\frac{1}{4}$.

3ª. $3ffh+hhh=-hg-3fgg$, in qua poſito valore ipſarum f & g, invenitur $h=\frac{1}{64h}$.

4ª. $3ffi+hhi=-hh-g^3-6fgh+1$, & ſubſtitutis valoribus jam inventis loco f, g, h, habetur $i=\frac{131}{512h^2}$.

5ª. $3ffl+hhl=-hi-6fgi-3ggh-3fhh$, ex qua eodem modo eruitur valor $l=\frac{509}{16384h^3}$.

Porro his valoribus ſubſtitutis in ſerie $fa^0+ga^1+ha^2+ia^3$ &c. habetur $x=h-\frac{1}{4}a+\frac{1}{64h}a^2+\frac{131}{512h^2}a^3+\frac{509}{16384h^3}a^4$ &c. Et poſito $a=1$, erit $x=h-\frac{1}{4}+\frac{1}{64h}+\frac{131}{512h^2}+\frac{509}{16384h^3}$.

Schol. I. *Si æquatio data duas incognitas* x *&* y *contineat, ut ſæpe in Geometria compoſita contingit, tunc ter-*

termini ſeriem componentes deſumuntur ex incognita, quæ minor eſt, ut ſeries convergens fiat, quemadmodum in Sch. Prop. præc. *dictum eſt. Ut ſi* x *major quam* y, *fiat ſeries* $x = ay^0 + by^1 + cy^2 + dy^3 + ey^4 + fy^5$ *&c. Cetera peraguntur, ut ſupra. At plura his addere non eſt hujus loci.*

SCHOL. II. *Cum ſeries infinitæ primo a* Mercatore Holſato per diviſiones, *ſecundo* a Cl. Newtono per radicum extractiones *quæſitæ fuerint, ut in* Schol. 2. Propoſ. 11. Cap. 3. *innuimus; demum celeberrimus* Leibnitius (a) *ad eas pervenit* per ſuppoſitionem ipſius ſeriei quæſitæ tanquam inventæ; *ita ut coefficientes teminorum ejuſdem ſeriei poſtea determinentur, ut ſeries ipſa determinata fiat: quæ ſane methodus commodior eſt longeque ceteris univerſalior, cum hæc non modo in calculo communi, ſed etiam in Differentiali & Integrali uſum habere poſſit plurimum. Nos facile & ad diſcentium intelligentiam aptum ſpecimen in hac propoſitione duntaxat exhibuimus.*

CAPUT XII.

De Geometrica Conſtructione Æquationum.

GEometricam æquationum conſtructionem aggredimur. Intelligent hinc tyrones ingentem Analyſeos in Geometria uſum. Intelligent qua ratione Euclides, Apollonius, magnus Archimedes aliique veteres

(a) Acta Erudit. Lipſiæ an. 1683.

Geometræ, quæ per Analyſin problemata ſubtiliter invenerant, ea deinde, ſuppreſſo calculo, compoſuerint ac ſynthetice demonſtraverint. Horum veſtigiis inſiſtentes *Simplices* tantum & *Quadraticas Æquationes* in hoc capite conſtruimus. Nam quæ altioris ſunt gradus, curvarum doctrinam ad conſtructionem requirunt. Præmittimus autem his quantitates Analyticas Geometrice exprimendi rationem, cum id tyronibus ſoleat negotium faceſſere.

PORISMA I.

Fractiones in terminos proportionales reſolvere.

FRactiones Analyticæ, quibus incognita æqualis exiſtit, in terminos proportionales reſolvuntur variis quidem modis.

I. Sit $x = \frac{ab}{c}$, erit $c \cdot a :: b \cdot x$, nimirum eſt x quarta proportionalis *per Prop.* 12. *l.* 6. *Eucl.*

II. Sit $x = \frac{aa - bb}{c - d}$, quia $a + b \times a - b = aa - bb$, erit $c - d \cdot a + b :: a - b \cdot x$ *per. Propoſ. cit.*

III. Sit $x = \frac{ab}{c} + \frac{adc}{n}$. Inveniatur $\frac{ab}{c} = n$, & $\frac{adc}{n} = m$, *per Propoſ. cit.*; erit $x = n + m$.

IV. Sit $x = \frac{ab + cd}{m + n}$, quia nulla litera in numeratore bis habetur, fiat $a \cdot d :: c \cdot$ ad quartam proportionalem $= r$,

$=r$, erit $ar=cd$ *per Prop.* 16. *l.* 6. *Eucl.* proinde æquatio mutatur in hanc $\frac{ab+ar}{m+n}$, in qua reperitur bis a, adeoque ſic reſolvitur $m+n \cdot b+r :: a \cdot x$.

V. Sit $x=\frac{aabc}{ddf}$, fiat primo $d \cdot a :: a \cdot$ ad quartam proportionalem $=m$, erit $aa=dm$; proinde ponendo dm loco aa, erit $x=\frac{dmbc}{ddf}$, ſeu $x=\frac{mbc}{df}$. Secundo fiat $d \cdot m :: b$ ad quartam proportionalem $=n$, erit $dn=mb$; unde ſurrogando dn loco ipſius mb, erit $x=\frac{dnc}{df}$, ſeu $x=\frac{nc}{f}$, itaque habetur $f \cdot n :: c \cdot x$, ut ſupra *num.* I. aut ſi fiat $f \cdot n :: c \cdot l$, erit $x=l$, cum ſit $\frac{nc}{f}=l$, ut patet.

COROLL. *Hinc facile invenitur recta æqualis pluribus fractionibus datis. Nam ſit* $x=\frac{ab}{c}+\frac{adc}{n}+\frac{cn}{f}$ *&c. ſi reperiatur, ut ſuperius factum eſt*, per Prop. 1?. l. 6. $\frac{ab}{c}=n$, $\frac{adc}{n}=m$, & $\frac{cn}{f}=l$, *erit* $x=n+m+l$, *nempe* $x=$ *rectæ lineæ compoſitæ ex rectis* n, m, l &c.

PORISMA II.

Summam, differentiam quadratorum, & radicum extractiones Geometrice exprimere. Fig. 10.

I. SIt triangulum ABC rectangulum in B, cujus hypothenuſa ſit $AC = a$, unum latus $AB = b$, & alterum $BC = c$, quia $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ *per* 47. *l.* 1. *Eucl.* erit $aa = bb + cc$, & $AC\,(a) = \sqrt{bb + cc}$. Similiter quia $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2$, erit $bb = aa - cc$, & $AB\,(b) = \sqrt{aa - cc}$. Item $cc = aa - bb$, & $BC\,(c) = \sqrt{aa - bb}$.

II. Quod ſi demittatur perpendiculum BD, ita ut baſis ſegmentum ſit $DC = x$, erit $AD = a - x$, & ob angulum rectum in D, erit $\overline{BD}^2 = \overline{BC}^2 - \overline{DC}^2 = cc - xx$, unde $BD = \sqrt{cc - xx}$. Similiter quia $\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = bb - aa + 2ax - xx$, erit $BD = \sqrt{bb - aa + 2ax - xx}$.

III. Similiter (*Fig.* 11.) ſi in ſemicirculo ABC ducantur duæ quælibet chordæ AB & BC, erit angulus ABC rectus *per* 20. *l.* 3. *Eucl.* Hinc poſita diametro $AC = a$, $AB = b$, & $BC = c$, habentur eædem omnino quantitates, quæ ſupra. Erit enim $AC\,(a) = \sqrt{bb + cc}$. Item $AB\,(b) = \sqrt{aa - cc}$, & $BC\,(c) = \sqrt{aa - bb}$.

IV.

IV. Demiſſo autem ex quovis circuli puncto perpendiculo $BD = y$, ſit $DC = x$, erit $AD = a - x$, & *per Coroll. Propoſ.* 17. *l.* 6. *Eucl. ex Tacquet* $\overline{BD}^2 = AD \times DC$, hoc eſt $yy = ax - xx$, unde $BD\ (y) = \sqrt{ax - xx}$. Vel ſit $ED = x$, $DB = y$, & CE, vel $BE = a$, erit $\overline{BD}^2 = \overline{EB}^2 - \overline{ED}^2$ hoc eſt $yy = aa - xx$, & $BD\ (y) = \sqrt{aa - xx}$.

COROLL. I. *Ex his patet quomodo exprimi poſſit Geometrice hæc, aut alia ſimilis quantitas Analytica* $\frac{r}{n}\sqrt{bb + cc}$. *Nam inveniatur ut ſupra*, $a = \sqrt{bb + cc}$, *fiet* $\frac{r}{n}\sqrt{bb + cc} = \frac{ar}{n}$ *quæ quantitas ſi* per Poriſ. 1. *in analogiam reſolvatur, haberi poteſt* $\frac{ar}{n} = l$, *unde* $\frac{r}{n}\sqrt{bb + cc} = l$.

COROLL. II. *Patet quo pacto media proportionalis per quantitatem radicalem deſignata, Geometrice ſit exprimenda. Nam* (Fig. 11.) *ſi fiat* AD $= a$ *&* DC $=$ b, *& ducatur* BD *perpendicularis ad diametrum* AC, *erit* BD $= \sqrt{ab}$ per Cor. Prop. 17. l. 6. Eucl. ex Tacquet.

COROLL. III. *Patet demum qua ratione dato plano* $=$ ab *reperiatur quadratum illi æquale* cc. *Nam ſit, ut prius*, AD $=$ a, DC $=$ b, *& ducatur in circulo ad diametrum* AC *perpendicularis* DB $=$ c, *erit* $\overline{DB}^2 = AD \times DC$ per Cor. cit. *hoc eſt* cc $=$ ab. *Contra vero dato quadrato* cc,

& alterutro plani bd *latere*, *facile eſt invenire ipſum planum.*

PORISMA III.

Æquationes ſecundi gradus Geometrice conſtruere.

QUatuor formulis comprehendi ſolent omnes ſecundi gradus æquationes; earumque radices ob duplicem valorem affirmativum, & negativum duplici ſigno afficiuntur $\pm$ ut ſæpe diximus, ſcilicet

	Formulæ	*Radices.*
1.	$x^2 = ax + bb.$	$x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + b^2}$
2.	$x^2 = -ax + bb$	$x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + b^2}$
3.	$x^2 = ax - bb$	$x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b^2}$
4.	$x^2 = -ax - bb$	$x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b^2}$

I. Pro prima formula conſtruenda (*Fig.* 12.) fiat triangulum *ABC* rectangulum in *B*, cujus latus $BC = b$, & $AB = \frac{1}{2}a$, erit *per Poriſ.* 2. baſis $AC = \sqrt{\frac{1}{4}aa + bb}$. Centro igitur *A* & intervallo *AB* fiat circulus *ABE*, & producatur *AC* in *D*, erit recta $CD = x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$, ut patet.

Vel ſic (*Fig.* 13.) fiat $AB = \frac{1}{2}a$, & perpendicularis $BC = b$; tum centro *A* cum intervallo baſis *AC* fiat ſemicirculus *DCE*, in quo producatur *AB* hinc inde ad peripheriam, erit $DB = x$.

Nam

Nam $DB = AB + AD$ vel AC, ſed $AB = \frac{1}{2}a$, & $AC = \sqrt{\frac{1}{4}aa + bb}$; ergo $DB = x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$.

II. Pro ſecunda formula conſtructio eſt eadem: nam in *Fig.* 12. (non producta AC in D) erit $CE = x$. Nam $CE = CA - AE$, ſeu AB: ſed $AB = \frac{1}{2}a$, & $CA = \sqrt{\frac{1}{4}aa + bb}$, ergo $CE = \sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a = x$.

Vel in *Fig.* 13. $EB = AE$, (vel AC) $- AB$, proinde $EB = \sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a = x$.

III. Pro tertia formula conſtruenda (*Fig.* 14.), quæ quidem duas habet radices poſitivas, ut ſignorum diſpoſitio docet *per num.* 3. *Prop.* 1. *Cap.* 6. ſit in ſemicirculo AEB diameter $AB = a$, erit ſemidiameter AC, vel CB, vel $CE = \frac{1}{2}a$. Elevetur ex puncto B perpendicularis $BF = b$, & ducatur diametro AB parallela EF, quæ ſecabit circulum in E. Tum ex puncto E ducta ED ad diametrum perpendiculari, erit $AD = x$.

Nam $AD = AC + CD$; ſed $AC = \frac{1}{2}a$, & $CD = \sqrt{\frac{1}{4}aa - bb}$, ergo $AD = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb} = x$.

Erit autem DB altera radix poſitiva. Nam $DB = BC - CD$, proinde $DB = \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb} = x$.

IV. Formula autem quarta, cujus conſtructio eſt omnino eadem, ambas radices habet negativas, ut ex ſignorum diſpoſitione dignoſcitur *per num.* 3. *Prop.* 1. *Cap.* 6. quarum prima erit $-AD = -AC - CD$, hoc eſt $-\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb} = x$. Altera vero $-DB = -CB + CD$, hoc eſt $-\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb} = x$.

SCHOL.

Schol. I. *Ceterum ad abſolutam problematis reſolutionem una radix & quidem poſitiva ſufficit. Proinde Carteſius de radicibus negativis conſtruendis ſolicitus nunquam fuit. Recentiores tamen ſingulas conſtruunt, ut & nos in tertia & quarta formula. Pro prima autem formula radix negativa* (Fig. 13.) *erit* EB. *Nam* $EB = -AE$ (*ſeu* $-AC$) $+ AB$; *proinde* $EB = \frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb} = x$. *Pro ſecunda formula radix negativa erit* $-DB$. *Nam* $-DB = -BA - AD$ (*ſeu* $-AC$) *ideſt* $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb} = x$. *Hinc apparet conſtructionem eſſe eandem, ſed radices negativas ad partem oppoſitam eſſe deſumendas; hoc eſt ſi radix poſitiva* DB *deſumpta fuit ex* B *verſus* D, *e contrario radix negativa* BE *deſumi debet ex* B *verſus* E: *quæ regula in Geometricis conſtructionibus ſemper obſervatur.*

Schol. II. *Si in tertia vel quarta formula* BF $(= b)$ *major ſit, quam* CB $(= \frac{1}{2}a)$ *fieri non poterit, ut parallela* EF *circulum ſecet, & in hoc caſu problema erit impoſſibile. Ratio eſt, quia ubi* $b > \frac{1}{2}a$ *erit quoque* $bb > \frac{1}{4}aa$, *ideoque* $\sqrt{\frac{1}{4}aa - bb}$ *fit quantitas imaginaria, & problema contradictionem involvere ſignum eſt. Quod ſi* $BF = CB$, *hoc eſt* $b = \frac{1}{2}a$, *tunc parallela* EF *non ſecat circulum, ſed illum tangit in puncto, ex quo ſi ducatur perpendicularis* ED, *hæc in centro conſiſtit, & recta* CD *evaneſcit, proinde* $\sqrt{\frac{1}{4}a^2 - bb} = 0$. *Hæc ex ipſa conſtructione patent.*

Schol. III. *Ut obtineantur æquationes ſupra allatæ, vel aliæ ſimiles, propoſito problemate, delineatur figura, in*

in qua problema ipſum quaſi ſolutum ponitur, ducendo in ea lineas perpendiculares, parallelas, angulos æquales, circulos, triangula ſimilia, vel rectangula, aliaſque figuras, quarum notæ relationes & proprietates Algebraicas æquationes nobis exhibent. Quæ quidem plenius docent problemata, quæ ſequuntur, fere omnia ab Euclide, Archimede, aliiſque ſummis Geometris de induſtria accepta: cum in his primo verſari ex uſu diſcentium eſſe putaverim.

PROBL. I.

Datam rectam AB *ſectam utcunque in* C *ita producere in* E, *ut rectangulum* AEB *ſit æquale quadrato* CE.

FActum jam ſit (*Fig.* 15.) & eſto data $AB = a$, $CB = b$, quæſita $BE = x$, erit tota $AE = a + x$, & ex conditione problematis $AE \times EB = \overline{CE}^2$, hoc eſt

$$ax + xx = b^2 + 2bx + xx$$

$$ax = b^2 + 2bx$$

$$ax - 2bx = b^2$$

Div. per $a - 2b$ $\quad x = \dfrac{b^2}{a - 2b}$.

Qua æquatione in terminos proportionales reſoluta *per Poriſ.* 1. habetur $a - 2b.\ b :: b.\ x.$ Eſt ergo $BE = x$ tertia proportionalis, unde oritur conſtructio, quæ ſequitur.

Conſtructio (*Fig.* 16.) Sumatur in *AB* pars $CD = CB$, erit $DB = 2b$, & $AD = a - 2b$. Fiat AD ($a - 2b$).

DC (*b*) :: *DC*, vel *CB* (*b*). *BE* (*x*). tertiam proportionalem quæſitam.

Demonſtr. Cum ex conſtruct. ſit *AD*. *DC*, vel *CB*:: *CB*. *BE*, erit componendo *per Propoſ.* 18. *l.* 5. *AC*. *DC*, vel *CB* :: *CE*. *BE*. Eſt autem ut *AC* antecedens ad *CB* conſequentem, ita *CE* antecedens ad *BE* conſequentem, ergo *AE* ſumma antecedentium erit ad *CE* ſummam conſequentium, ut *CE* una antecedentium ad *BE* unam conſequentium *per Prop.* 12. *l.* 5. ergo rectangulum ſub extremis $AE \times BE$, nempe *AF*, erit æquale quadrato ſub mediis $CE \times CE$, id eſt *CH per Prop.* 17. *l.* 6. Quod erat &c.

SCHOL. *Hinc apparet methodus, aut alia huic affinis, quam* Euclides, Archimedes, Apollonius *aliique veteres in ſuis problematibus inveſtigandis, conſtruendis, ac demonſtrandis tenuerunt.*

PROBL. II.

In dato triangulo ABC *quadratum inſcribere.* Fig. 17.

ESto factum, & quadratum inſcriptum ſit *EDFG* Ducatur perpendicularis *AH*, quæ ob triangulum datum erit pariter data. Sit $BC=a$, $AH=b$, & *HL* ſeu $FG=x$, erit $AL=b-x$. Jam ob triangula ſimilia *BAC* & *FAG* erit *AH* (*b*). *BC* (*a*) :: *AL* ($b-x$). *FG* (*x*), proinde

$$bx=ab-ax$$
$$bx+ax=ab$$

Divid. per $a+b$ $\quad x=\dfrac{ab}{a+b}$

Reſo-

Reſoluta æquatione in terminos proportionales, habetur $a+b. a :: b. x$ *per Poriſ.* 1. unde patet quæſitam $LH = x$ eſſe quartam proportionalem. Hinc oritur conſtructio, quæ ſequitur.

Conſtructio. Producta *BC* indefinite, fiat $HM = BC$, & $MN = AH$, erit $HM + MN = a + b$. Jungatur *AN*, cui ex puncto *M* parallela ducatur *LM*, quæ ſecabit *AH* in puncto quæſito *L*.

Demonſtr. Ob parallelas *AN* & *LM* triangula *AHN*, & *LHM* ſunt ſimilia, proinde *MN*, ſeu *AH. AL* :: *HM*, ſeu *BC. LH*. Item ob triangula ſimilia *BAC* & *FAG* eſt *AH. AL* :: *BC. FG*; ergo *AH. AL* :: *BC. LH*, ideoque *BC* ad duas *FG* & *LH* eandem rationem habet, quæ idcirco ſunt æquales *per Propoſ.* 9. *lib.* 5. Eſt ergo *EDFG* quadratum. Quod &c.

COROLL. *Tam ex hoc, quam ex præc. problemate apparet, fractiones, quibus quantitas incognita in æquatione finali æquatur, in terminos proportionales eſſe reſolvendas, ut in* Poriſ. 1. *docuimus.*

PROBL. III.

***In quadrilatero* ABDE *circulo inſcripto rectanguli, quod fit ex diagonalibus* AD × EB *ad rectangula, quæ fiunt ex lateribus oppoſitis* AE × BD, *&* AB × DE, *rationem invenire.* Fig. 18.**

DUcatur *DF* faciens angulum *EDF* æqualem angulo *ADB*, erunt triangula *ADB* & *EDF* ſimilia, cum etiam anguli *DAB* & *BED* ſint æquales, utpote eidem

arcui *BD* inſiſtentes, proinde *AD*. *DE* :: *AB*. *EF*; & ſi ponatur $AD = a$, $DE = b$, $AB = c$, & $EF = x$, erit $a.\ b :: c.\ x$, adeoque $ax = bc$.

Similiter triangula *ADE* & *BDF* ſunt æquiangula & ſimilia. Nam ſi æqualibus ex conſtructione angulis *EDF*, & *ADB* addatur communis *ADF*, erit $ADE = BDF$. Item $DAE = DBF$, cum eidem arcui *DE* inſiſtant, & hinc *BF*. *AE* :: *BD*. *AD*. Jam ſi ponatur tota $BE = f$, erit $BF = f - x$. Sit $AE = d$, & $BD = e$, erit $f - x.\ d :: e.\ a$; ideoque $af - ax = ed$. Addatur huic æquationi altera $ax = bc$, erit $af = ed + bc$; unde patet rectangulum, quod fit ex diagonalibus $AD \times BE = af$, æquari duobus rectangulis, quæ fiunt ex lateribus oppoſitis $AE \times BD = ed$, & $AB \times DE = bc$ ſimul ſumptis.

Demonſtr. Ob triangula ſimilia *ADB* & *EDF* eſt *AD*. *DE* :: *AB*, *EF*: ergo *per* 16. *l*. 6. $AD \times EF = DE \times AB$. Item ob triangula ſimilia *ADE* & *BDF* eſt *BF*. *AE* :: *BD*. *AD* ergo *per Prop. cit.* $AD \times BF = AE \times BD$. Sed $AD \times BF$ & $AD \times EF = AD \times EB$ totam *per* 1. *l*. 2.; ergo $AD \times EB = DE \times AB \times AE \times BD$. Quod erat &c.

Schol. *Præclarum hinc oritur* Ptolomæi (*a*) *theorema*: In quadrilatero inſcripto in circulo rectangulum ſub diagonalibus æquale eſt rectangulis, quæ a lateribus oppoſitis fiunt. *Quod quidem magnum habet in Trigonometria uſum.*

PROBL.

(*a*) Almageſti lib. 1. Cap. 9.

PROBL. IV.

Datam rectam AB *media & extrema ratione ſecare.* Fig. 19.

SEcta ſit AB in puncto C, ut imperatur; & ſit ipſa $AB = a$, $AC = x$, erit $CB = a - x$, & per conditionem problematis cum tota AB debeat eſſe ad majus ſegmentum AC, ut AC ad ſegmentum minus CB, erit $a.\ x :: x.\ a - x$. hinc

$$xx = aa - ax$$
$$xx + ax = aa$$

Prop. 1. *Cap.* 8. $\quad \frac{1}{4}a^2 \quad \frac{1}{4}a^2$

$$x^2 + ax + \frac{1}{4}a^2 = \frac{5}{4}a^2$$

Extra. rad. $\quad x + \frac{1}{2}a = \sqrt{\frac{5}{4}a^2}$

$$x = -\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$$

Conſtruct. A rectæ AB puncto B elevetur perpendicularis $BD = \frac{1}{2}a$; ductaque AD, ſecetur $ED = DB$ $(= \frac{1}{2}a)$. Facto deinde centro in A cum intervallo AB, ſecetur $AC = AE$, erit C punctum quæſitum.

Demonſtr. Triangulum ABD eſt rectangulum in B, ergo $\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2 = aa + \frac{1}{4}aa = \frac{5}{4}a^2$; unde $AD = \sqrt{\frac{5}{4}a^2}$. Eſt autem $DE = DB = \frac{1}{2}a$; ergo $AE = AD - DE = \sqrt{\frac{5}{4}a^2} - \frac{1}{2}a = AC$.

Inſuper $CB = a - x$, erit $= \frac{3}{2}a - \sqrt{\frac{5}{4}a^2}$. Sunt autem $AB = a$, $AC = \sqrt{\frac{5}{4}a^2} - \frac{1}{2}a$, & $CB = \frac{3}{2}a - \sqrt{\frac{5}{4}a^2}$

 tales

tales inter ſe, ut rectangulum ſub extremis $AB \times CB$ ſit æquale quadrato mediæ AC, hoc eſt $\frac{3}{2}a^2 - a\sqrt{\frac{5}{4}a^2} = \frac{3}{2}a^2 - a\sqrt{\frac{5}{4}a^2}$, hinc AB, AC & CB ſunt continue proportionales *per* 17. *l.* 6. proinde AB ſecta eſt in C media & extrema ratione.

Coroll. I. *Hinc facile eruitur ſynthetica demonſtratio. Nam ſi data* AB *ſecetur in* C, *ita ut rectangulum* $AB \times CB$ *ſit æquale quadrato* AC, *ut docet* Prop. 11. l. 2. *erit* AB. AC : : AC. CB per 17. lib. 6. *ergo* AB *ſecta eſt in* C *ſecundum mediam & extremam rationem* per Defin. 3. lib. 6.

Schol. *Certum eſt quadratum* $x^2 + ax + \frac{1}{4}a^2$ *poſſe oriri tam ex radice poſitiva* $x \times \frac{1}{2}a$, *quam ex negativa* $- x - \frac{1}{2}a$. *In primo caſu habetur radix poſitiva, quam ſupra conſtruximus* $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$; *in ſecundo radix negativa & æqualis nihilo* $x = -\frac{1}{2}a - \sqrt{\frac{5}{4}a^2}$, *quæ tamen conſtrui poteſt hac ratione*:

Suppoſita ſuperiori conſtructione, producatur BA *verſus* F, *& fiat* $AF = DB = \frac{1}{2}a$, *&* $FG = AD = \sqrt{\frac{5}{4}a^2}$, *erit tota* $AG = \frac{1}{2}a + \sqrt{\frac{5}{4}a^2} = -x$, *&* $GB\ (= a - x) = \frac{3}{2}a + \sqrt{\frac{5}{4}a^2}$. *Sunt porro tres continue proportionales* $BA = a$, $AG = \frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$, *&* $GB = \frac{3}{2}a + \sqrt{\frac{5}{4}a^2}$. *Nam* $BA \times GB = \overline{AG}^2$, *cum utrinque oriatur* $\frac{3}{2}a^2 + a\sqrt{\frac{5}{4}a^2}$.

Coroll. II. *In conſtructione radicum negativarum quantitates cognitæ ſigno — affectæ deſumi debent ad partem oppoſitam radicum poſitivarum; quod* in Schol. 1. Poriſ. 3. *fuit prænotatum, & ex hac conſtructione apparet,*

PROBL.

PROBL. V.

Datam rectam AB *inæqualiter ita ſecare in* C, *ut quadrata totius AB, & minoris ſegmenti* BC *ſint tripla quadrati, quod fit a ſegmento majori* AC. Fig. eadem.

Sit $AB = a$, $AC = x$, erit $CB = a - x$, & per conditionem problematis

$$a^2 + a^2 - 2ax + x^2 = 3x^2$$
$$2a^2 - 2ax = 2x^2$$
$$x^2 + ax = a^2$$

Prop. 1. *Cap.* 8. $\quad \frac{1}{4}a^2 \quad \frac{1}{4}a^2$

$$x^2 + ax + \frac{1}{4}a^2 = \frac{5}{4}a^2$$

Extr. rad. $\quad x + \frac{1}{2}a = \sqrt{\frac{5}{4}a^2}$

$$x = -\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$$

Quæ quidem æquatio cum ſit eadem ac ſuperioris problematis, indicio eſt rectam AB ſecandam eſſe media & extrema ratione, ut duo quadrata $\overline{AB}^2$ & $\overline{BC}^2$ ſint tripla quadrati $\overline{AC}^2$. Facta igitur eadem conſtructione, ſectaque AB in C media & extrema ratione, erunt tres continue proportionales $AB = a$, $AC = \sqrt{\frac{5}{4}a^2} - \frac{1}{2}a$, & $BC = \frac{3}{2}a - \sqrt{\frac{5}{4}a^2}$; atque hinc quadrata $\overline{AB}^2 + \overline{BC}^2 = 3\overline{AC}^2$, hoc eſt $\frac{9}{2}a^2 - 3a\sqrt{\frac{5}{4}a^2}$.

COROLL.

Coroll. *Hinc oritur Theorema* 4. *l.* 13. *Euclidis*: Si recta lineâ media & extrema ratione ſecta fuerit, totius & minoris portionis utraque ſimul quadrata tripla ſunt quadrati ejus, quod a majori fit portione. *Cujus demonſtratio ex ſuperiori æquatione & conſtructione ſponte ſua ſequitur.*

PROBL. VI.

A dato puncto E rectam ducere, quæ datum circulum tangat. Fig. 20.

Punctum *E* poſitione datur, circulus vero *ABGD* poſitione & magnitudine; proinde ducta *AE*, dantur *AD* & *DE*. Sit ergo $AD = a$, $DE = b$ & $EG = x$; producta autem *AD* in *B*, erit tota $BE = 2a + b$, & $BE \times ED = \overline{EG}^2$ *per* 36. *l.* 3 *Eucl.* hinc habetur æquatio

$$2ab + b^2 = x^2$$

$$x = \sqrt{b^2 + 2ab}$$

Conſtr. Centrum circuli *A* & punctum datum connectantur recta *AE*, ſuper qua deſcribatur ſemicirculus *AEG*, in quo ducantur chordæ *AG* & *EG*.

Demonſtr. Angulus *AGE* in ſemicirculo rectus eſt *per Prop.* 31. *l.* 3. ergo $\overline{AE}^2 = \overline{AG}^2 + \overline{EG}^2$ *per* 47. *l.* 1. hoc eſt $a^2 + 2ab + b^2 = a^2 + x^2$, ſeu $x^2 = 2ab + b^2$; proinde $x = \sqrt{2ab + b^2} = EG$.

Verum

Verum demonſtratio ſynthetica unico verbo poteſt abſolvi. Nam angulus *AGE* in ſemicirculo eſt rectus; ergo *EG* circulum tangit in *G per* 16. *l.* 3. Quod erat &c.

PROBL. VII.

Dato circulo invenire latus trianguli æquilateri in eodem circulo inſcribendi. Fig. 21.

SIt $AB = x$ latus trianguli quæſiti, & ducatur chorda BC æqualis ſemidiametro $CD = a$, quæ erit latus hexagoni *per* 15. *l.* 4. *Eucl.* eritque $AC = 2a$. Ob angulum *ABC* in ſemicirculo rectum $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$, hoc eſt

$$xx + a^2 = 4a^2$$
$$xx = 3a^2$$
$$x = \sqrt{3a^2}$$

Eſt ergo latus trianguli æquilateri medium proportionale inter circuli, cui inſcribitur, radium, ejuſque triplum, nempe inter a & $3a$. Hinc ſi ponatur $a = 1$, erit trianguli æquilateri latus ad radium ut $\sqrt{3}$ ad 1.

Conſtructio tamen facilior eſt, quæ vulgo traditur. (*Fig.* 22.) Super diametro *AB* deſcribatur triangulum æquilaterum *ABC*, & ducatur *CD*, erit *CD* latus quæſitum. Nam in triangulo rectangulo *CDB* erit $CB = 2a$, $DB = a$, & $CD = x$, unde $x = \sqrt{3aa}$.

COROLL. *Hinc infertur, non omnes concinniores conſtructiones Geometricas ex calculo Analytico derivari, nec proinde*

inde eidem ſemper eſſe inhærendum, cum in ſolutione problematum Geometricorum elegantiora quandoque media operantis ingenium, quam talis determinata æquatio ſuppeditet, quæ quidem reſpicit problema illud ſolitarie ſumptum & ab omnibus aliis independens, ut Cl. Wolfius *obſervat, & ex hoc ipſo problemate intelligi poteſt: quod quidem multis aliis modis eleganter & a ſuperiori æquatione independenter conſtruitur.*

Schol. *Ex penultima æquatione habetur egregium theorema, quod eſt Prop.* 12. *lib.* 13. *Euclidis:* Si in circulo triangulum æquilaterum deſcribatur, quadratum ex trianguli latere triplum eſt ejus, quod fit ex circuli ſemidiametro. *Nam ſi ponatur* BC = a, *&* AC = 2a, *erit* $\overline{AC}^2 - \overline{BC}^2 = \overline{AB}^2$, *proinde* $\overline{AB}^2 = 3a^2$, *&* $\overline{AB}^2 . \overline{BC}^2 :: 3a^2 . a^2 :: 3 . 1.$ *adeoque* BC, *ſeu* CD $= \sqrt{\tfrac{1}{3}}$. Fig. 21.

PROBL. VIII.

Rationem invenire, quam pentagoni latus habet ad hexagoni & decagoni ſimul ſumpta latera in eodem circulo inſcriptorum. Fig. 23.

SIt *AC* latus pentagoni, diviſoque bifariam arcu *AC* in *B*, chorda *AB*, ſeu *BC* erit latus decagoni, & circuli radius *AF*, vel *FC* latus hexagoni, ut patet. Ducatur *FD* perpendiculariter ſuper *AB*, quam bifariam ſecabit in *D per* 3. *l.* 3. & ducatur *EB*. Duo triangula *ACF*, *ECF* ſunt æquiangula & ſimilia; nam præter angulum communem *C*, angulus *CFE* = *FAC*: eſt enim arcus *BC* gr. 36.

& *BD*

& *BD*, gr. 18. ergo totus *DC*, nempe angulus *CFE*, gr. 54. Similiter cum in trigono iſoſcele *AFC* angulus ad centrum ſit gr. 72., erunt ſinguli ad baſim gr. 54. proinde $FAC = CFE$.

Sit $AF = a$, $AC = b$, $EC = x$, erit $AE = b - x$, $AB = c$. Ratione triangulorum ſimilium *ACF*, *ECF* eſt $AC. AF :: AF. EC$, hoc eſt $b. a :: a. x$, hinc $bx = aa$. Deinde triangulum iſoſcele *AEB* ſimile eſt triangulo iſoſceli *ABC*. Habent enim angulum *A* communem, proinde omnes alios æquales, ut patet. Sunt ergo proportionales $AE. AB :: AB. AC$, hoc eſt $b - x. c. :: c. b$, unde $bb - bx = cc$. Addatur huic altera æquatio, habetur $bb = aa + cc$, hoc eſt latus pentagoni regularis poteſt latera hexagoni & decagoni eidem circulo inſcriptorum, ut in *Prop.* 10. *lib.* 13. propoſuit Euclides.

P R O B L. IX.

Dato quadrato ABCD, *ex angulo* A *ducere rectam* AE, *ut pars* FE *a latere* DC *continuato intercepta ſit æqualis datæ rectæ* M. Fig. 24.

EX puncto *E* demittatur *EG* perpendicularis ad *AE*, quæ lateri producto *AB* occurrat in *G*. Ducta perpendiculari *EH*, facile erit oſtendere *EG* æqualem eſſe *AF*: cum triangula *EHG* & *ABF* ſint rectangula, ſimilia per *Prop.* 8. *lib.* 6. & etiam æqualia: & $EH = CB = AB$, proinde *EG* & *AF* latera homologa & æqualia.

Sit ergo *AB*, vel $BC = a$, $FE = M = b$, $AF = EG = y$, & $BG = x$, erit $AE = y + b$, & $AG = a + x$.

Tum ob triangula ſimilia *ABF* & *AEG*, erit *AB*. *AF* :: *AE*. *AG*, hoc eſt $a . y :: y + b . a + x$; unde habetur $aa + ax = yy + by$. Deinde ob triangulum *AEG* rectangulum in E, $\overline{AG}^2 = \overline{AE}^2 + \overline{EG}^2$, hoc eſt $a^2 + 2ax + x^2 = 2y^2 + 2by + bb$. In hac ſecunda æquatione ponatur pro $y^2 + by$ ejus valor ex prima inventus $aa + ax$, habebitur $aa + 2ax + x^2 = 2a^2 + 2ax + b^2$, quæ reducitur ad ſimpliciſſimam ſecundi gradus $xx = aa + bb$, atque $x = \sqrt{aa + bb}$, $= BG$.

Conſtr. (*Fig.* 25.) Producatur quadrati latus *AD* in *O*, ita ut *AO* ſit æqualis rectæ datæ *M*, erit $\overline{BO}^2 = \overline{AB}^2 + \overline{AO}^2$ (ob triangulum rectangulum *OAB*) $= aa + bb$, & BO $= \sqrt{aa + bb} = x$. Producatur ergo latus *AB* in *G*, ita ut *BG* ſit æqualis *BO*, & ſuper *AG* diametro deſcribatur ſemicirculus *AEG*, qui ſecabit quadrati latus *DC* productum in *E*, & conjungatur *AE*, erit *EF* recta quæſita.

Schol. *Hæc conſtructio eadem eſt, quæ a Pappo fuit allata, & a Carteſio indicatur, tum ad uſum reductionis æquationum oſtendendum, tum ad docendum nos plurimum intereſſe, unam vel aliam quantitatem incognitam aſſumere. Sive enim pro incognita aſſumatur* BF, *ſive* FC, *vel* AF *aut* CE, *ſemper oritur æquatio biquadratica. Sumpta autem* BG, *prodit æquatio ſimpliciſſima ſecundi gradus, ut vidimus. Jure igitur in hoc ipſo problemate ſolvendo poſt Carteſium recentiores Analyſtæ fere omnes ingenium ſuum exercuerunt, apud quos illud paſſim invenias.*

PROBL.

PROBL. X.

Circulum invenire datæ ſuperficiei conicæ æqualem. Fig. 26.

SIt circuli quæſiti radius $= x$, & ratio radii ad peripheriam ſit $r.p$. Si fiat $r.p :: x.\frac{px}{r}$, erit $\frac{px}{r}$ peripheria circuli; & inde innoteſcit area circuli $= \frac{px^2}{2r}$ *per Coroll.* 1. *Propoſ.* 5. *Archim. ex Tacquet.*

Sit coni dati latus $= a$, peripheria circuli baſis ejuſdem $= p$, erit ſuperficies conica $= \frac{1}{2}\ ap$ *per Coroll.* 1. *Prop.* 13. *ex Archim. cit.* Erit ergo ex hypotheſi $\frac{px^2}{2r} = \frac{1}{2}\ ap$, & multiplicando per $2r$.

$$px^2 = arp$$
$$x^2 = ar$$
$$x = \sqrt{ar}$$

COROLL. *Hinc oritur Archimedis theorema*: Circulus, cujus radius eſt medius proportionalis inter coni recti latus & radium baſis conicæ, æqualis eſt ſuperficiei conicæ. *Eſt ejuſdem Prop.* 13. *de Sphæ. & Cylin.*

Demonſtr. (*Fig.* 26. & 27.) Sit conus rectus datus *R*. Fiat circulus *MLN*, cujus radius *OL* ſit medius proportionalis inter coni latus *AB*, & conicæ baſis radium *BC*: hoc eſt ponatur radius *OL* medius proportionalis inter *AB*

 & *BC*;

& *BC*; erit *per Prop.* 7. *Archim. cit.* peripheria *BED* ad peripheriam *MLN* ut *BC* ad *OL*, ſive ut *OL* ad *AB*: proinde rectangulum ſub prima, nempe peripheria *BED* & quarta *AB*, ſive ejus dimidium (nempe triangulum, cujus baſis eſt peripheria *BED* & quarta *AB* altitudo) æquatur rectangulo ſub ſecunda, ſeu peripheria *MLN* & tertia *OL*, ſeu triangulo ſub eadem ſecunda *MLN* & tertia *OL*, quod eſt ejuſdem rectanguli dimidium. Sed triangulum, cujus baſis eſt peripheria *BED*, & altitudo *AB* æquatur dati coni ſuperficiei *per Coroll.* 1. *Propoſ.* 13. *ex Archim. cit.* & triangulum ſub peripheria *MLN* & tertia *OL* æquatur circulo *OLM per Propoſ.* 5. *ex Archim cit.* ergo habetur circulus æqualis conicæ ſuperficiei quæſitus. Quod &c.

PROBL. XI.

Dato cono recto ABC, *circulum invenire æqualem ſuperficiei conicæ* BDCE *planis parallelis* BC & DE *interceptæ.* Fig. 28.

SEctus ſit conus per axem triangulo *ABC*, erunt rectæ *BC* & *DE* parallelæ *per Prop.* 16. *l.* 11. *Eucl.* cum ſint ſectiones communes ejuſdem trianguli cum planis parallelis *BHC* & *DEL*. Sit coni latus $AB = l$, $AD = m$, erit $BD = l - m$: item eſto *BF* radius circuli baſis $= r$, *DG* radius circuli paralleli $= s$, & circuli quæſiti radius $= x$. Ob triangula ſimilia *ABC* & *ADE* eſt *AB*. *BF* :: *AD*. *DG*, hoc eſt $l.\ m :: r.\ s$, atque hinc $ls = mr$, & $ls - mr = 0$.

Superficies autem conica *ABC* æquatur circulo (*per Theor*

Theor. præc.) cujus radius $= \sqrt{lr}$, ſeu cujus radii quadratum $= lr$: item ſuperficies conica *ADE* æquatur circulo, cujus radius $= \sqrt{ms}$, ſeu cujus radii quadratum $= ms$; erit ergo radius circuli quæſiti, ſive ejus quadratum

$$x^2 = lr - ms$$

$$x = \sqrt{lr - ms}$$

COROLL. *Si ſuperior æquatio* $x^2 = lr - ms$ *in proportionem reſolvatur, erit* $\overline{l-m}.x :: x.\ \overline{r+s}$. *Nam* $\overline{l-m} \times \overline{r+s} = lr + ls - mr - ms$. *Inventum autem fuit ſupra* $ls - mr = 0$, *proinde* $\overline{l-m} \times \overline{r+s} = lr - ms$; *atque hinc oritur theorema Archimedis, quod eſt Propoſ.* 15. *de Sphær. & Cylin.* Circulus habens radium proportione medium inter interceptam parallelis planis *BC* & *DE* partem lateris *DB* $(= l - m)$ & ſummam radiorum *BF* + *DG* $(= r + s)$ circulorum, qui ſunt in planis parallelis, æquatur ſuperficiei conicæ *BDCE* parallelis planis interceptæ.

PROBL. XII.

Dato cylindro ſphæram æqualem invenire.

Fig. 29. & 30.

SUpponatur factum, & ſphæra *M* ſit æqualis cylindro *ABCD*. Concipiatur ſphæræ *M* circumſcriptus cylindrus *EFGH*, habens tam latus *FH*, quam diametrum baſis *HG*, utrumque æquale diametro ſphæræ *LN*; erit cylindrus

lindrus *EFGH* ſphæræ circumſcriptus ejuſdem ſeſquialter; hoc eſt $= \frac{3}{2}$ ſphæræ *per Propoſ.* 32. *Archim. ex Tacquet.* Proinde ſi fiat $PC = \frac{3}{2}\ DC$, erit cylindrus $BP = \frac{3}{2}$ cylindri *BD per Prop.* 14. *l.* 12. & æqualis cylindro circumſcripto *EFGH*, cum ſphæra & cylindrus *BD* ex hypotheſi ſint æquales.

Sit jam cylindri dati diameter $BC = a$, altitudo $DC = b$, erit altitudo $PC = \frac{3}{2}\ b$. Sit autem ſphæræ diameter *LN*, vel $FH = x$.

In cylindris æqualibus *BP* & *GP* erit baſis *BCK* ad baſim *CHZ* (vel quadrata diametrorum earundem $\overline{BC}^2$, $\overline{GH}^2$ *per Prop.* 2. *l.* 12.) ut reciproce altitudo *FH* ad altitudinem *PC*, hoc eſt

$$a^2 . x^2 :: x . \tfrac{3}{2}\ b$$
$$x^3 = \tfrac{3}{2}\ a^2\ b$$
$$x = \sqrt[3]{\tfrac{3}{2}\ a^2\ b}$$

Ex quo patet problema eſſe ſolidum, & ad ejus ſolutionem duas medias proportionales eſſe inveniendas inter a & $\frac{3}{2}\ b$; nempe $\div\div\ a,\ x,\ \frac{x^2}{a},\ \frac{x^3}{a^2}$. Nam ex quadratis diametrorum a^2 & x^2 habentur tres continue proportionales $\div\div\ a,\ x,\ \frac{x^2}{a}$ *per Prop.* 20. *l.* 6. *Eucl*, ex ſuperiori autem æquatione $x^3 = \frac{3}{2}\ a^2\ b$ eruitur $\frac{x^3}{a^2} = \frac{3}{2}\ b$ dividendo per a^2, proinde $\div\div\ a,\ x,\ \frac{x^2}{a},\ \frac{3}{2}\ b$.

Schol.

SCHOL. *Egregium hoc Archimedis problema; quod ipſe ſtatim initio* lib. 2. de Sphæ. & Cyl. *Analytico fere modo propoſuit, illud deinde componit per duas medias proportionales, & nos eodem modo, ſed aliquanto brevius abſolvemus.*

Demonſtr. Sit $CD = \frac{2}{3} CP$, erit cylindrus $BD = \frac{2}{3} BP$ cylindri. Tum inter BC & CP inveniantur duæ mediæ proportionales LN & R per modum aliquem mechanicum ex allatis a P.Tacquet ad *Prop. 13. lib. 6. Eucl.* Dico LN eſſe diametrum ſphæræ æqualis cylindro dato BD. Nam ſphæræ M concipiatur circumſcriptus cylindrus $EFGH$, cujus baſis, diameter, & altitudo ſint æquales diametro ſphæræ LN, vel EG, erit eadem ſphæra $= \frac{2}{3}$ cylindri circumſcripti $EFGH$ *per Propoſ. 32. Archim. ex Tacquet.* Idem autem cylindrus circumſcriptus æquatur cylindro BP. Nam cum ſint quatuor continue proportionales BC, LN, R, CP, erit viciſſim $BC . R :: LN . CP$ *per 16. lib. 5. Eucl.* Sed $BC . LN :: \overline{BC}^2 . \overline{LN}^2$ *per 2. lib. 12. Eucl.* ergo erit BC ad LN, baſis ad baſim, ut LN vel EG altitudo cylindri GF ad CP altitudinem cylindri BP; ergo hi duo cylindri æquantur *per 15. l. 12. Eucl.* proinde etiam ſphæra M & cylindrus BD æquabuntur, cum ambo ſint $= \frac{2}{3}$ æqualium cylindrorum BP, GF. Quod &c.

APPENDIX

De conſtructione problematum ſolidorum.

Cum toties in hoc tractatu de reſolutione problematum ſolidorum per curvas obtinenda mentio facta

Uu ſit,

ſit, abs re non erit hoc loco de illa breve aliquod ſpecimen exhibere. Cum autem nulla ſit methodus, quantum ego opinor, Carteſiana brevior, aut facilior, quæ nimirum circulum duntaxat & parabolam adhibet, hanc ipſam ex Cl. Halleji (*a*) annotatis explicare conabimur, præmiſſis, quæ ad rei intelligentiam viam ſternunt. Hæc autem methodus ſecundum terminum ex æquatione ſublatum petit.

DEFINITIONES.

I. *Curva Algebraica* illa dicitur, quæ per æquationem Algebraicam definiri poteſt; vel illa, in qua per lineas rectas explicari poteſt ratio, quam ſingula curvæ puncta ad axem, vel diametros dicunt. Sic in circulo *AMB* (*Fig. 31.*) ex quovis diametri puncto *P* ducta perpendiculari *PM*, ponatur $AB = a$ & $AP = x$, erit $BP = a - x$; ſit $PM = y$, erit $\overline{MP}^2 = AP \times PB$ *per Propoſ. 35. lib. 3. Eucl.*, hoc eſt $yy = ax - xx$; & cum talis æquatio, ſeu circuli proprietas ſemper eadem reperiatur ex quocunque diametri puncto ducatur perpendicularis *PM*, dicitur *æquatio circuli*, & ipſe circulus *curva Algebraica*, licet ob ſummam in ejus deſcriptione facilitatem tanquam figura plana conſideretur. Perpendicularis *PM* dicitur *ſemiordinata*, ſeu *applicata* ad axem, vel ad diametrum *AB*; portio vero diametri *AP* dicitur *abſciſſa*. Utraque quantitas dicitur *variabilis*, quia utraque creſcit & decreſcit, adeoque exprimi ſolent per *x* & *y*. *Semidiameter* vero circuli, & in parabola aliisque curvis *latus rectum*,

(*a*) In Tranſact. Anglicanis n. 183. p. 335.

rectum, ſeu *parameter*, dicuntur quantitates *conſtantes*: nam aliis creſcentibus, vel decreſcentibus, ipſæ eædem manent.

II. *Parabola* eſt curva Algebraica (*Fig.* 32.) in qua ſemiordinatæ quadratum æquatur facto ex abſciſſa in parametrum. Sit parabola *BAC*, cujus vertex punctum *A*, recta *AD* indefinite producta *axis*, *AL latus rectum*, ſeu *parameter*: perpendiculares ad axem *PM*, *pm*, dicuntur *ſemiordinatæ*; *AM*, vel *Am abſciſſæ*. Quælibet vero recta *ON*, vel *PQ* axi *AD* parallela dicitur *Parabolæ diameter*.

Ponatur ſemiordinata $PM = x$, & abſciſſa $AM = y$, parameter $AL = a$, cum ex *Propoſ. 11. lib. 1. Conic. Apollonii* ſit $\overline{MP}^2 = AM \times AL$, erit $xx = ay$: quæ quidem eſt proprietas & natura parabolæ, quæ per talem æquationem Algebraicam deſignatur.

CoROLL. *Eſt ergo* $x = \sqrt{ay}$, *hoc eſt ſemiordinata* MP *eſt media proportionalis inter parametrum* AL *&* *abſciſſam* AM. *Nam* a . x :: x . y, *unde* $x = \sqrt{ay}$.

PORISMA I.

Parabolam in plano deſcribere. Fig. 33.

Sint in eodem plano duæ rectæ *AZ* & *BX* indefinite productæ, & ad angulos rectos ſibi invicem occurrentes in *A*. Sumatur in *AZ* parametri longitudo *AD*, & in *BX* axis *AB*. Tum ex puncto *D* ducatur *DE* axi *AB* parallela. Poſtea ſumantur aliæ duæ rectæ *AR* & *CT*, quarum prior *AR* circa punctum fixum *A* revolvatur,

altera vero *CT*, ſervato ſemper ad axim *AB*, vel ad diametrum *DE* ſitu parallelo, progrediatur in recta *AZ*, quæ rectæ tamen ſic moveantur, ut jugiter ſit *DF* = *AC*. Nam ſi notentur puncta interſectionum duarum illarum rectarum *AR* & *CT*, prodibit curva quæſita, quam dico eſſe parabolam.

Demonſtr. Ex aliquo curvæ puncto *M* ducatur recta *MP* parallela rectæ *AD* & ad axem *AB* ordinata, erit *MP* = *AC*, & *CM* = *AP* *per Prop. 44. l. 1. Eucl.* Jam vero ob triangula ſimilia *FAD* & *MAC* eſt *AD* . *DF* (ſeu *AC* ex conſtructione) : : *AC* . *CM*, ſeu *AP*. Habetur ergo *AC*, ſeu *PM* (= x) media proportionalis inter *AD* parametrum (= a) & abſciſſam *AP* (= y), proinde $x^2 = ay$; ideoque curva *AFL* eſt parabola *ex Defin. 2. hujus.*

Schol. *Ad praxim loco rectarum* AR & CT, *quas modo explicato moveri concipimus, apponi ſolent duæ regulæ mobiles, quarum prima circa clavum in* A *fixum revolvatur, altera in ſitu ad axem parallelo progreditur, ex quarum mutua interſectione habentur puncta pro deſcriptione parabolæ, ut Fig. 34. ſatis oſtendit.*

PORISMA II.

Parabolam in plano alia ratione deſcribere.

Fig. 35.

Parabolæ parameter *AL* producatur indefinite in *N*, & ex puncto *A*, quod erit parabolæ vertex, elevetur perpendicularis indefinita *AO*. Deinde ſumptis in *NL* centris quotcunque pro arbitrio (circino tamen usque ad

ad *L* aperto) fiant arcus rectam *AO* interſecantes in punctis *M*, *M*, *M*, tum etiam rectam *AN* in punctis *P*, *P*, *P*. Patet quamlibet ex his *AM* eſſe mediam proportionalem inter datam parametrum *AL* & abſciſſas *AP*, *AP*, *AP per Coroll. Prop. 13. l. 6. Eucl. ex Tacquet.* Igitur poſitis pro parametro *AL*, & *AQ* pro axe parabolæ deſcribendæ, ſi abſciſſæ *AP*, *AP*, *AP* transferantur in axem *AQ* in punctis 1, 2, 3 &c. & ex iiſdem punctis ducantur ad axem *AQ* normales 1*N*, 2*N*, 3*N* &c. quæ æquales ſint ipſi *AM*, *AM*, *AM* &c. curva per extrema harum perpendicularium tranſiens erit parabola *BAC*. Quæ quidem parabolæ deſcriptio ceteris longe tutior eſſe cenſetur, cum mechanicis inſtrumentis, quæ plerumque vitio laborant, minus ſit obnoxia. Innititur autem *Coroll. poſt Defin. 2.* ut patet.

PROPOSITIO I.

Explicatur conſtructionis Carteſianæ methodus. Fig. 36.

1. Deſcribatur parabola *OAN per Poriſ. 1. aut 2.* cujus vertex *A* & axis *AQ*, parameter autem *AL* ponatur $a = 1$, & transferatur parametri dimidium $\frac{1}{2}a = \frac{1}{2}$ ex vertice *A* in *C*.

2. Ex puncto *C* ſecetur $CD = \frac{1}{2}p$ in ipſo axe deorſum, ſi in æquatione habeatur $-p$; vel ſurſum retrogrediendo verſus *A*, ſi fuerit $+p$, & producendo (ſi opus ſit) axem ad partes ipſius *A*. Quod ſi punctum $p = 0$, hoc eſt ſi æquatio tertio termino careat, tunc *CD* non ducitur.

3. Ex puncto D (vel ex C, ſi deſit CD) erigatur ad axem perpendicularis $ED = \frac{1}{2} q$ dextrorſum quidem ſi fuerit in æquatione $- q$ (ut in hac figura) at ſiniſtrorſum ſi fuerit $+ q$, erit EA radius circuli deſcribendi.

4. Centro E cum intervallo AE deſcribatur circulus, qui parabolam ſecat in punctis O, B, N, ex quibus ducta ad axem perpendicula OP, BC & FN dant radices æquationis quæſitas; hoc eſt, quæ ſunt ad axis dextram (ut FN) ſemper ſunt veræ, quæ vero ad ſiniſtram ut BC, & OP ſunt falſæ. Hæ ſimul ſumptæ radicem veram FN æquare debent, cum æquatio cubica ſecundo termino careat, ut ſupponitur. At, quæ ſequuntur exempla rem clare oſtendunt.

Schol. *Ratio hujus methodi a locorum Geometricorum doctrina pendet, quæ docet modum combinandi circulum cum parabola, ceteriſque ſectionibus Conicis, ſublatis incognitarum ſecundis terminis. Sed* Carteſius *rem diſſimulavit, ſola praxi contentus. Nos digitum ad fontem tantummodo intendimus, ne propoſitæ appendicis limites transgrediamur.*

PROPOSITIO II.

Æquationum Cubicarum conſtructio exemplis illuſtratur. Fig. 37.

I. Conſtruenda ſit æquatio $x^3 - px - q = 0$. Deſcribatur parabola *per Poriſ. 1. vel 2.* OAN, cujus parameter AL ſit $a = 1$, vertex A, & axis AQ. Sumatur $AC = \frac{1}{2} a = \frac{1}{2}$, & $CD = \frac{1}{2} p$. Deinde elevetur ex

ex puncto D perpendicularis $ED = \frac{1}{2}q$ dextrorſus quidem ob $-q$; & centro E cum intervallo EA deſcribatur circulus OAN, qui ſecabit parabolam in punctis O, B, & N, ex quibus ducta perpendicula OP, BC, FN dant propoſitæ æquationis radices, duas quidem falſas OP & BC ad ſiniſtram axis, & tertiam FN veram ad dexteram, quæ duabus prioribus æquatur.

Demonſtr. Sit $FN = x$, ut ſupponitur; abſciſſa $AF = y$, & parameter $AL = 1$, erit ex natura parabolæ $\overline{FN}^2 = AF \times AL$, hoc eſt $x^2 = 1y$ *per Defin. 2.* unde $AF = x^2$, & FD (ſeu EM) $= AF - AC - CD = x^2 - \frac{1}{2} - \frac{1}{2}p$. Item $MN = FN - FM$ (ſeu DE) $= x - \frac{1}{2}q$, adeoque in triangulo rectangulo EMN erit $\overline{EN}^2 = \overline{EM}^2 + \overline{MN}^2 = x^4 - px^2 - qx + \frac{1}{2}p + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{4}$.

Similiter in triangulo rectangulo ADE erit $\overline{AE}^2 = \overline{AD}^2 + \overline{DE}^2 = \frac{1}{2}p + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{4}$. Ex radiorum autem AE, EN æqualitate habetur æquatio, proinde deletis utrinque terminis ſimilibus remanet.

$$x^4 - px^2 - qx = 0$$
$$x^3 - px - q = 0$$

Quod ſi ſumatur aliqua ex radicibus falſis v. g. $OP = -x$, eadem manet demonſtratio. Nam ſit abſciſſa $AP = y$, ſemiordinata $OP = -x$, parameter $AL = 1$, erit ex natura parabolæ *per Defin. 2.* $\overline{OP}^2 = AP \times AL$, hoc eſt $x^2 = 1y$; unde $AP = x^2$. Eſt autem $PT = DE$ $(= \frac{1}{2}q)$; unde

unde $OT (= OP + PT) = -x + \frac{1}{2}q$, & $AD = \frac{1}{2} + \frac{1}{2}p$; proinde $DP (= DA - AP) = \frac{1}{2} + \frac{1}{2}p - x^2$.

Jam in triangulo rectangulo ETO eſt $\overline{EO}^2 = \overline{OT}^2 + \overline{ET}^2$, ſeu $\overline{EO}^2 = x^4 - px^2 - qx + \frac{1}{2}p + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{4}$. Item in triangulo rectangulo EDA eſt $\overline{AE}^2 = \overline{AD}^2 + \overline{DE}^2 = \frac{1}{2}p + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{4}$, adeoque ob radiorum EO & AE æqualitatem habetur æquatio; deletis proinde utrinque terminis ſimilibus, remanet

$$x^4 - px^2 - qx = 0$$
$$x^3 - px - q = 0.$$

II. Sit æquatio conſtruenda $x^3 + px - q = 0$, ſeu $x^3 + 3x - 6 = 0$. Deſcripta parabola (*Fig.* 38.) NAM, cujus axis AO, vertex A. Aſſumpta parametro $= 1$, ſecetur $AC = \frac{1}{2}$, & ſumatur $CD (= \frac{1}{2}p) = \frac{3}{2}$ ſupra punctum A, cum adſit in æquatione $+ p$. Elevata deinde perpendiculari ad axem $ED (= \frac{1}{2}q) = 3$, dextrorſum ob $-q$, fiat centrum in E, & intervallo EA deſcribatur circulus EAM, qui parabolam ſecat in unico puncto M, ex quo ducta normalis PM ad axem dat radicem veram PM.

Demonſtr. Producta PM in F, cui occurrat perpendicularis EF demiſſa ex puncto E, quæ parallela & æqualis erit ipſi DP, ut patet; ſit $AC = \frac{1}{2}$, $CD = \frac{1}{2}p$, $ED = PF = \frac{1}{2}q$, $PM = x$, $AP = y$, & parameter $= 1$. Erit $FM (= PF - PM) = \frac{1}{2}q - x$, & $DA (= DC - AC) = \frac{1}{2}p - \frac{1}{2}$.

Jam

Jam ex natura parabolæ $\overline{PM}^2 = 1 \times AP$, hoc eſt $xx = 1y$, unde $AP = xx$, & $PD\ (= PA + AD) = xx + \frac{1}{2}p - \frac{1}{2}$. Ob duo triangula rectangula EDA, EFM eſt $\overline{AE}^2 = \overline{ED}^2 + \overline{AD}^2 = \frac{1}{4}qq + \frac{1}{4}pp - \frac{1}{2}p + \frac{1}{4}$. Similiter $\overline{EM}^2 = \overline{EF}^2 + \overline{MF}^2 = x^4 + px^2 - qx + \frac{1}{4}qq + \frac{1}{4}pp - \frac{1}{2}p + \frac{1}{4}$. Sed EA & EM ſunt æquales, ergo & eorum quadrata, proinde habetur æquatio, & deletis terminis ſimilibus utrinque, remanet

$$x^4 + px^2 - qx = 0$$
$$x^3 + px - q = 0$$

Surrogatis autem valoribus p, q, erit $x^3 + 3x - 6 = 0$.

CoROLL. *Ubi circulus parabolam non ſecat (ut in hoc caſu) niſi in unico puncto, ſignum eſt alias duas æquationis datæ radices eſſe imaginarias. Quod optime convenit cum iis, quæ diximus in* Propoſ. 4. Cap. 9. *nimirum duas radices imaginarias haberi, ſi in æquatione fuerit* +p.

III. Conſtruenda ſit æquatio $x^3 + px + q = 0$, ſeu $x^3 + 1x + 1 = 0$. Deſcribatur parabola (*Fig.* 39.) cujus parameter $= 1$, vertex A, axis AR. Secetur $AC = \frac{1}{2}$, & a puncto C verſus A ſumatur $CD = \frac{1}{2}p = \frac{1}{2}$. Punctum D cadit præciſe in A, & ex puncto A, vel D ducatur perpendicularis $AE = \frac{1}{2}q = \frac{1}{2}$ ſiniſtrorſus ob $+q$, factoque centro in E cum intervallo EA, fiat circulus EAF ſecans parabolam in F, ex qua interſectione ducta ad axem normali FM, habetur radix falſa FG (cum ſit ad axis ſiniſtram) & aliæ duæ imaginariæ *ex ſuper. Corol.*

Demonſtr. Sit parameter $AL = 1$, abſciſſa $AC = y$, & applicata FC ex hypotheſi $= - x$, erit ex natura parabolæ $\overline{FC}^2 = AC \times AL$, hoc eſt $x^2 = 1y$, unde AC, ſeu EN (quæ ex centro E ducitur ad FC perpendicularis) $= x^2$, & ED, ſeu $NC = \frac{1}{2}$; ideoque $FN = (FC - NC) = - x - \frac{1}{2}$. Jam igitur in triangulo rectangulo ENF eſt $\overline{EF}^2 = \overline{EN}^2 + \overline{FN}^2$, hoc eſt

$$\begin{aligned} \tfrac{1}{4} &= x^4 + x^2 + 1x + \tfrac{1}{4} \\ x^4 &+ x^2 + 1x = 0 \\ x^3 &+ x + 1 = 0 \end{aligned}$$

Schol. *Si circulus parabolam tangit, duæ interſectiones coincidunt perinde ac illam in duobus punctis perexiguo ac fere nullo intervallo diſtantibus ſecaret, & ſectiones illæ in puncto contactus coirent; adeoque tunc æquatio duas habet radices æquales. Quod ſi eam nec tangit, nec ſecat, radices omnes ſunt impoſſibiles.*

PROPOSITIO III.

Æquationes biquadraticas conſtruere. Fig. 40.

Deſcripta parabola cum parametro, vertice & axe omnino ut *in præc. Prop.* factum eſt, debet inſuper pro æquatione biquadratica conſtruenda augeri, vel minui circuli deſcribendi radius AE; hoc eſt ſi in æquatione fuerit $- r$, additur; ſi fuerit $+ r$, ſubtrahitur ex quadrato radii AE rectangulum ar factum ex parametro a & ex

ex data quantitate r. Nam circuli hujus interſectiones cum parabola dant quæſitas æquationis radices, veras quidem ſemper ad axis dexteram, falſas ad ſiniſtram. Res fiet exemplis clariſſima.

I. Conſtruenda ſit æquatio $x^4 - qx - r = 0$, quæ præter ſecundum etiam tertio termino caret. Deſcripta parabola FAG, cujus latus rectum $AL = a = 1$, vertex A, axis AN, ex puncto C dextrorſum (ob $-q$) elevetur normalis $CE = \frac{1}{2} q$. Circuli deſcribendi radius AE augeri debet ob $-r$ rectangulo ar. Igitur producatur AE hinc inde indefinite, & AB ſecetur æqualis parametro $AL = a$, $AS = r$. Tum deſcripto ſuper BS ſemicirculo BDS, & ex puncto A elevata perpendiculari AD ad diametrum BS, erit $\overline{AD}^2 = BA \times AS = ar$, ductaque ED, erit $\overline{ED}^2 = \overline{AE}^2 + \overline{AD}^2$. Habetur ergo DE circuli deſcribendi radius quæſitus auctus rectangulo ar; factoque centro E cum intervallo ED, fiat circulus DRG, qui parabolam ſecabit in punctis R & G, ex quibus ducta ad axem perpendicula RM & NG dant radices æquationis propoſitæ, NG veram utpote ad axis dexteram, & MR falſam ad ſiniſtram. Ducatur ex puncto E perpendicularis EQ.

Demonſtr. Sit $NG = x$, & abſciſſa $AN = y$. $\overline{AD}^2 = AB \times AS = ar$, unde $AD = \sqrt{ar}$. $QG = CE - NG = \frac{1}{2} q - x$; & ex natura parabolæ $\overline{NG}^2 = NA \times a$, nempe $x^2 = ay$, atque hinc $y = \frac{xx}{a} = AN$. Similiter EQ $(= AN - AC) = \frac{xx}{a} - \frac{1}{2} a$.

Jam in triangulo rectangulo ECA eſt $\overline{AE}^2 = \overline{AC}^2 + \overline{CE}^2 = \frac{1}{4}qq + \frac{1}{4}aa$. Item in triangulo rectangulo EQG eſt $\overline{EG}^2 = \overline{QG}^2 + \overline{EQ}^2 + \frac{x^4}{a^2} - qx + \frac{1}{4}qq + \frac{1}{4}aa$. Tandem in triangulo EAD eſt $\overline{DE}^2 = \overline{AE}^2 + \overline{AD}^2 = \frac{1}{4}qq + \frac{1}{4}aa. + ar.$ Igitur ob æqualitatem radiorum ED & EG erit

$$ar = \frac{x^4}{a^2} - qx$$

$$x^4 - a^2 qx = a^3 r$$

(ob $a = 1$) $$x^4 - qx = r$$

$$x^4 - qx - r = 0$$

Idem demonſtrari poteſt aſſumpta radice negativa $RM = - x$.

II. Sit (*Fig.* 41.) conſtruenda æquatio $x^4 - 5x^2 + 2x + \frac{5}{2} = 0$, ſeu $x^4 - px^2 + qx + r = 0$. Deſcribatur parabola, cujus latus rectum $a = 1$, vertex A, & axis AD, in quo ſecetur $AC = \frac{1}{2}$ & $CD = \frac{1}{2}p$. Tum ſiniſtrorſum ob $+ q$ elevetur perpendicularis ad axem $DE = \frac{1}{2}q$. Liquet ex dictis minuendum eſſe ob $+ r$ circuli deſcribendi radium rectangulo ar. Proinde producta utrinque AE indefinite, ſecetur $AB = a$, & $AH = r$: tum ſuper diametro BH deſcribatur ſemicirculus HMB, & ex puncto A elevetur perpendicularis AM. Similiter ſuper AE deſcribatur ſemicirculus ANE, in quo ducatur chorda $= AM$; ſeu (quod idem eſt) ſecetur arcus AN poſito circino in A cum intervallo AM. Igitur intervallum EN erit radius quæſi-

quæſitus circuli centro E deſcribendi. Nam $\overline{EN}^2 = \overline{AE}^2 - \overline{AN}^2$ in triangulo rectangulo ENA, & $\overline{AN}^2 (= \overline{AM}^2) = BA \times AH = ar$. Itaque deſcripto ſic circulo, parabola interſecatur in punctis G, F, P & Q, ex quibus demiſſæ ad axem normales Gc, FR, OP & CQ, habentur quatuor æquationis datæ radices, duæ quidem ad axis dexteram veræ, & duæ falſæ ad ſiniſtram.

Demonſtr. Sumatur quælibet radix v. g. $OP = x$, & ſit abſciſſa $AO = y$, erit ex natura parabolæ $\overline{OP}^2 = AO \times a$, hoc eſt $x^2 = ay$, atque hinc $y = \frac{x^2}{a} = AO$. Deinde $\overline{AM}^2$, ſeu $\overline{AN}^2 = AB \times AH = ar$, unde $AN = \sqrt{ar}$, & in triangulo rectangulo ENA (ductis rectis EN & AN) erit $\overline{EN}^2 = \overline{AE}^2 - \overline{AN}^2 = \frac{1}{4}a^2 + \frac{1}{2}ap + \frac{1}{4}p^2 + \frac{1}{4}q^2 - ar$.

Similiter quia $PK = x + \frac{1}{2}q$, & OD, ſeu $EK = AO - AC - CD = \frac{x^2}{a} - \frac{1}{2}a - \frac{1}{2}p$, in triangulo rectangulo EKP erit $\overline{EP}^2 = \overline{EK}^2 + \overline{PK}^2 = \frac{x^4}{a^2} - \frac{px^2}{a} + \frac{1}{2}ap + \frac{1}{4}a^2 + \frac{1}{4}p^2 + qx + \frac{1}{4}q^2$. Igitur propter radiorum EP & EN æqualitatem habetur æquatio

$$\frac{x^4}{a^2} - \frac{px^2}{a} + qx = -ar$$

$$x^4 - apx^2 + a^2qx + a^3r = 0$$

Hinc

Hinc poſito $a = 1$, & ſurrogatis valoribus p, q & r, erit $x^4 - 5x^2 + 2x + \frac{5}{4} = 0$, qualis propoſita fuit.

CO ROLL. *Conſtructiones æquationum tertii & quarti gradus fere omnino conveniunt. In hoc tamen differunt, quod in cubicis circulus tranſit per parabolæ verticem, non autem in biquadraticis.*

SCHOL. I. *Præclara hæc* Carteſii *methodus id incommodi nonnullis habere viſa eſt, quod ſecundum æquationis terminum, ſi adſit, tolli jubeat. Huic moleſtiæ* Thomas Bakerus (*a*) Anglus *occurri poſſe animadvertit, ſi loco axis parabolæ, uti* Carteſius *fecit, adhiberetur aliqua parabolæ ipſius diameter, ad quam ducerentur ex interſectionibus parabolæ & circuli ea perpendicula, quæ ut vidimus, dant radices quæſitas. Hac regula, quam a ratione tradita circuli centrum determinandi,* Centralem *vocat, æquationes omnes cubicas & biquadraticas quomodolibet affectas & completas ſine ulla prævia reductione conſtruere docet. At vero tot illa nodis intricata eſt, & tot cautionibus de ſignis + & — obnoxia, ut remoto libro, vix poſſit quis omnes memoria retinere, ut* Cl. Hallejus (*b*) *teſtatur, qui multis annotationibus ei plurimum lucis attulit; quemadmodum deinde* Chriſtophorus Sturmius (*c*) & Wolfius (*d*) *fecerunt, apud quos videant, qui volunt.*

SCHOL. II. *Sicuti parabolam & circulum mutuo interſecari vidimus, & inde obtineri datarum tertii & quarti gradus æquationum radices, ita circulus cum ellipſi,* vel

(*a*) Clavis Geometrica Cathol. Londini 1684. (*b*) Tranſ. Anglic. n. 188 p. 135. an. 1687. (*c*) Matheſ. enucl. p. 349. (*d*) Conſtruct. æquat probl. 2;4

vel hyperbole, aut duæ quælibet ex iiſdem Conicis ſectionibus pari ratione combinari poſſunt ad problemata cujuſcunque gradus conſtruenda, quemadmodum Dn. de la Hire (*a*) & Marchio (*b*) Hoſpitalis *egregie fecerunt. Nos duas medias continue proportionales ex duarum parabolarum interſectione mox inveniemus. Menechmus id primus docuit.*

PROBLEMA I.

Inter datas a & q *duas medias proportionales invenire.* Fig. 42.

I. SIt prior quæſita $= x$, erunt quatuor continue proportionales $a \cdot x \cdot \frac{x^2}{a} \cdot \frac{x^3}{a^2}$, proinde $\frac{x^3}{a^2} = q$, & $x^3 = aaq$, ſeu $x^3 - aaq = 0$.

Vel ſic: ſit prior quæſita $= x$, altera $= y$. Quoniam ex hypotheſi $\div\div\; a \cdot x \cdot y \cdot q$ ſunt continue proportionales, erit $a^2 \cdot x^2 :: x \cdot q$ *per Coroll. Prop.* 20. *l.* 6. *Eucl.* adeoque $x^3 = aaq$, ſeu $x^3 - aaq = 0$, ut prius.

Pro conſtructione deſcripta parabola *FAP*, cujus parameter $a = 1$, vertex *A*, & axis *AQ*, ſecetur $AC = \frac{1}{2}$, & ex puncto *C* elevetur normalis $CE = \frac{1}{2}q$ dextrorſum quidem ob $-q$. Tum centro *E* cum intervallo *AE* deſcribatur circulus *EAP*, parabolam ſecans in puncto *P*, erit applicata *PM* radix quæſita $= x$.

Demonſtratio a ceteris non differt, ductis recta *EP* & perpendiculo *EN*. Nam ſi parameter $= a$, $AC = \frac{1}{2}a$,

EC

(*a*) Sections Coniques liv. 9. & 10. (*b*) Conſtruct. des æquat. Analyt.

$EC = \frac{1}{2}q$, $AM = y$, & $MP = x$, erit PN (= PM — MN) $= x - \frac{1}{2}q$. Cum autem ex natura parabolæ ſit $AM \times a = \overline{PM}^2$, hoc eſt $ay = xx$, erit $y = \frac{xx}{a}$, adeoque $AM = \frac{xx}{a}$, & MC, ſeu EN (= $AM - AC$) $= \frac{xx}{a} - \frac{1}{2}a$. Jam in triangulo rectangulo ENP eſt $\overline{EP}^2 = \overline{EN}^2 + \overline{PN}^2 = \frac{x^4}{aa} - qx + \frac{1}{4}aa + \frac{1}{4}qq$; eademque ratione in triangulo rectangulo ACE eſt $\overline{EA}^2 = \overline{AC}^2 + \overline{CE}^2 = \frac{1}{4}aa + \frac{1}{4}qq$, quæ duæ æquationes ſunt æquales ob æqualitatem radiorum ejuſdem circuli EP & AE, proinde æqualibus utrinque detractis, remanet

$$\frac{x^4}{aa} - qx = 0$$
$$x^4 - aaqx = 0$$
$$x^3 - aaq = 0$$

Coroll. *Hinc habetur modus inveniendi inter duas datas tot medias proportionales, quot volueris. Sint datæ* a *&* q, *&* *quæſitarum prima* = x. *Continuetur progreſſio Geometrica* $\div$ $a \cdot x \cdot \frac{x^2}{a} \cdot \frac{x^3}{a^2} \cdot \frac{x^4}{a^3} \cdot \frac{x^5}{a^4} \cdot \frac{x^6}{a^5}$ *&c. & numerus terminorum, qui binario ſuperat quæſitarum proportionalium numerum, fiat æqualis alteri quantitati datæ* q. *Sic ut habeantur tres mediæ proportionales, fiet* $\frac{x^4}{a^3} = q$, *unde erit*

de erit $x^4 = a^3 q$. *Item ſi quærantur quatuor proportione mediæ, fiat* $\frac{x^5}{a^4} = q$, *erit* $x^5 = a^4 q$ *&c ſic de ceteris.*

II. Sint, ut prius, datæ a & q, inter quas inveniendæ ſint duæ mediæ proportionales x & y, erit $a \cdot x :: x \cdot y$, unde $ay = x^2$. Similiter $x \cdot y :: y \cdot q$, adeoque $qx = y^2$ quæ duæ æquationes parabolæ naturam exprimunt *ex Defin. 2. hujus.* Ex prima habetur $y = \frac{x^2}{a}$; quo valore poſito in ſecunda, oritur $qx = \frac{x^4}{a^2}$, atque hinc $x^3 = aaq$, ſeu $x^3 - aaq = 0$, ut ſuperius.

Pro conſtruct. (*Fig.* 43.) ſecent ſe ad angulos rectos in *A* duæ rectæ *DB* & *EC*. Secetur $AB = a$ datarum priori, & $AC = q$ alteri datæ. Tum ſuper axe *AD* cum parametro *AB* deſcribatur parabola *AMP*. Item ſuper axe *AE* cum parametro *CA* deſcribatur parabola *AMN*; & ex punctis interſectionum ducantur ad ſuos axes ſemiordinatæ *MP* & *MN*, erunt *AN* & *AP* duæ mediæ proportionales quæſitæ, hoc eſt $AB \cdot AN :: AN \cdot AP :: AP \cdot AC$.

Demonſtr. Sit $AN (= PM) = x$, & $AP (= MN) = y$, æquatio ad parabolam *AMP* eſt $\overline{PM}^2 = AP \times AB$, hoc eſt $x^2 = ay$. Eadem ratione æquatio ad parabolam *MAN* eſt $\overline{MN}^2 = AN \times AC$, hoc eſt $y^2 = qx$. Cum igitur ex priori habeatur $a \cdot x :: x \cdot y$, ex altera vero $x \cdot y :: y \cdot q$, evidens eſt in continua eſſe ratione $a \cdot x \cdot y \cdot q$, nempe $AB \cdot AN \cdot AP \cdot AC$. Quod &c.

Coroll. *Ex inventione duarum proportione mediarum ſequitur celebratiſſima cubi duplicatio, quæ tandiu veterum Geometrarum mentem exercuit. Sint enim in continua proportione* a. x. y. q, *ſeu* (*poſitis* a = 1 *&* q = 2) 1. x. y. 2. *habebit prima* 1 *ad quartam* 2 *rationem triplicatam ejus, quam habet eadem prima* 1 *ad ſecundam* x per Defin. 10. lib. 5. Eucl. *Sed cubus ex prima* 1 *ad cubum ex ſecunda* x *habet pariter rationem triplicatam ejus, quam habet eadem prima* 1 *ad ſecundam* x, *hoc eſt quam habet* 1 *ad* 2 per Propoſ. 33. lib. 11. Eucl. *ergo cubus ex ſecunda* x *duplus eſt cubi dati* 1.

Schol. D. Guido Grandus (a) *celeberrimus in Academia Piſana Matheſèos Profeſſor Meſòlabium expeditiſſimum pro inventione duarum mediarum inter duas datas publicavit Florentiæ anno* 1728., *cujus ope non modo duplicatio cubi, ſed anguli quoque triſectio, diviſio ſphæræ in partes datæ rationis, & omne prorſus problema ſòlidum poteſt faciliime ſolvi.*

PROBLEMA II.

Datum angulum AED, *vel arcum* ABCD *in tres æquales partes dividere.* Fig. 44.

Supponatur diviſio jam facta in punctis *B* & *C*, erit chorda *AB* = *BC* = *CD*. Ducantur radii *EA*, *EB* *EC* & *ED*, tum chorda *AD*; at *BN* ſit parallela radio *CE*.

Triangula *AEB*, *BEC*, *CED* ſunt æqualia, iſoſcelia & ſimilia, quod evidens eſt. Item triangula *EAB* & *BAR* ſimi-

(a) Flores Geometrici in Appendice.

ſimilia ſunt: nam præter angulum *ABR* communem, angulo quoque ad centrum *AEB* æquatur angulus *BAD* ad circumferentiam *per Prop.* 20. *lib.* 3. *Eucl.* cum ſubtendat arcum duplum, unde *EB*, ſeu *EA*·*AB*:: *AB*·*BR*; & cum triangulum *AEB* ſit iſoſcele, erit quoque *BAR* iſoſcele, proinde $AB = AR$. Idem omnino demonſtrari poteſt de triangulo *DEC* reſpectu trianguli *CMD*, quod pariter eſt iſoſcele, unde $CD = MD$.

Jam in iſoſcele *BAR* anguli *ABR* & *ARB* ad baſim æquantur, itaque angulus *CBE* ($= ABR$) $= BRA$, ſeu *MRE* ad verticem, ideoque *BC* & *RM* ſunt parallelæ. At vero triangula *ABR* & *BRN* ſunt æquiangula: nam, præter angulum communem *BRN*, angulus quoque *NBR* = (ob parallelas *BN*, *CM*) $CEB = BEA = BAR$, & tertius $ABR = BNR$ tertio. Cum autem triangulum *ABR* ſit iſoſcele, erit quoque *NBR* iſoſcele; hinc $BN = BR$, & ob triangulorum ſimilitudinem, *AR*, ſeu *AB*·*BR*:: *BR*·*RN*. Erat autem *EA*·*AB*::*AB*·*BR*; ſunt ergo in continua ratione *EA*·*AB*·*BR*·*RN*.

Sit jam radius $EA = 1$ chorda $AB = x$, erunt quatuor termini continuo proportionales $1 \cdot x \cdot x^2 \cdot x^3$. At quia in iſoſcele *BAR* eſt $AB = AR$, item in iſoſcele *CDM* eſt $CD = MD$, in parallelogrammo autem *NC* eſt $BC = NM$, ſequitur chordam *AD* (quam voco q), addita *NR*, æqualem eſſe tribus rectis *AB*, *BC* & *CD*; hoc eſt $AD + NR = AB + BC + CD$, proinde $q + x^3 = 3x$. Habetur ergo æquatio $x^3 - 3x + q = 0$.

Conſtr. Deſcribatur parabola (*Fig.* 45.) *MAN*, cujus parameter *AL* ſit $a = 1$, vertex *A*, & axis *AQ*, in quo ſecentur $AC = \frac{1}{2} a = \frac{1}{2}$, & CD ($= \frac{1}{2} p$) $= \frac{3}{2}$. Tum ex pun-

ꝺto D elevetur normalis ad axem $DE = \frac{1}{2} q$ ad ſiniſtrum axis latus ob $+ q$. Factoque centro E cum intervallo EA fiat circulus $EANM$, ſecans parabolam in punctis P, N & M; ex quibus ad axem applicata PT eſt æqualis chordæ quæſitæ $AB = BC = CD$. At RN eſt radix æqualis chordæ AF, quæ tripartito dividit alterum arcum AFD. Tertia MQ eſt radix falſa duabus PT & RN veris æqualis.

Demonſtr. Sit abſciſſa $AT = y$, & ſemiordinata $TP = x$, parameter $AL = 1$; erit ex natura parabolæ $\overline{TP}^2 = AT \times AL$, hoc eſt $x^2 = 1y$, unde $AT = x^2$. Eſt autem $AD = AC + CD = \frac{1}{2} + \frac{3}{2} = 2$; & in triangulo rectangulo AED eſt $\overline{AE}^2 = \overline{AD}^2 + \overline{ED}^2 = 4 + \frac{1}{4} qq$.

Deinde DT ($= EG$) $= DA - AT = 2 - x^2$, & $GP = GT$ (ſeu ED) $+ TP = \frac{1}{2} q + x$, proinde in triangulo rectangulo EGP eſt $\overline{EP}^2 = \overline{EG}^2 + \overline{GP}^2 = 4 - 4x^2 + x^4 + x^2 + qx + \frac{1}{4} qq$, adeoque ob radiorum EP & AE æqualitatem oritur æquatio

$$4 + \tfrac{1}{4} qq = x^4 - 3x^2 + qx + 4 + \tfrac{1}{4} qq$$
$$x^4 - 3x^2 + qx = 0$$
$$x^3 - 3x + q = 0$$

Coroll. *Hinc liquet oleum & operam perdere Geometriæ tyrones, dum anguli triſectioni per lineam rectam & circulum inveniendæ iuſudant, cum ſit problema ſolidum.*

PROBL.

PROBLEMA III.

In æquationibus cubicis uſus præc. problematis peculiaris oſtenditur.

SIt æquatio cubica $x^3 - px - q = 0$, in qua cubus ex triente quantitatis cognitæ tertii termini major ſit quadrato ex ſemiſſe ultimi termini, hoc eſt $\frac{1}{27} p^3 > \frac{1}{4} qq$, tunc erunt tres radices reales & inæquales *per Propoſ.* 1. *num.* 2. *Cap.* 9., quæ quidem per Algebram obtineri non poſſunt, & occurrit caſus ille *irreducibilis* de quo diximus *in Schol.* 3. *Prop.* 9. *Cap.* 9. Adhibitis enim Cardani formulis, æquationis radix, quæ realis eſſe debet, per quantitates imaginarias exprimitur, quod eſt abſurdum. At vero per ſubtenſas arcuum, ſeu per anguli triſectionem æquationis ejuſdem radices opportune deſignantur modo nunc explicando. *Fig.* 44.

Radio $AE = \sqrt{\frac{1}{3} p}$, qui ſit proportione medius inter quantitatis cognitæ p trientem & unitatem (*Vide Schol. Probl.* 7. *Cap.* 12.) deſcribatur circulus ADF, in quo ducatur chorda $AD = \frac{3q}{p}$, quæ nempe ſit ad datam quantitatem q, ut 1 ad $\frac{1}{3}p$, adeoque obtinebitur, ſi fiat ut $\frac{1}{3}p. 1 :: q. \frac{3q}{p}$. Deinde ſupponatur diviſus arcus $ABCD$ in tres æquales partes AB, BC, CD *per Probl. præc.* ita ut AB, vel BC, aut CD ſit $= -x$: patet enim ex ſignis in data æquatione haberi duas radices falſas & unam

veram. Cum triangula AEB, BAR & RBN ſimilia ſint, erit AE ($\sqrt{\frac{1}{3}p}$). AB ($-x$) :: AB ($-x$). BR ($\frac{x^2}{\sqrt{\frac{1}{3}p}}$) :: BR ($\frac{x^2}{\sqrt{\frac{1}{3}p}}$). RN ($\frac{x^4}{-\frac{1}{3}px}$) $= -\frac{x^3}{\frac{1}{3}p}$. Eſt autem recta $AD + NR = AB + BC + CD$, ut *in præc. Probl.* vidimus, proinde $\frac{3q}{p} - \frac{x^3}{\frac{1}{3}p} = -3x$, atque hinc $\frac{x^3}{\frac{1}{3}p} = 3x + \frac{3q}{p}$, hoc eſt $x^3 = px + q$, ſeu $x^3 - px - q = 0$.

Deſcripta igitur parabola OAN (*Fig.* 37.) *per Poriſ.* 1. *aut* 2. tum etiam circulo ENO, qui eam ſecat in punctis O, B, N, ſi ex his applicentur ad axem rectæ BC, OP & FN, prodibunt radices æquationis datæ, nimirum BC æqualis chordæ $AB = -x$, OP æqualis chordæ $AF = -x$, & $FN = x$; quæ quidem eſt vera & duabus prioribus (ad ſiniſtram poſitis) falſis æqualis. Itaque caſus in Algebra *irreducibilis* ex Geometria per anguli triſectionem facile ſolvitur.

Schol. Albertus Girardus [a] *hanc methodum, cujus eſt auctor, fuſe demonſtrat, quam* Nic. de Martino [b] *in Neapolitano Lyceo Regius Mathematum profeſſor egregie contraxit.*

PROBL.

(a) Invention Nouvelle en l'Algebre an. 1629. Vid. Fr. à Schooten in Append. de Cubic. æquat. reſolut. (b) Nova Algeb. Elem. lib. 2. Cap. 7. num. VI.

PROBLEMA IV.

Quæſtionem Pappi reſolvere.

PAppus Alexandrinus initio *lib.* 7. *Collect. Mathemat.* mentionem facit de problemate quodam perplexo, ac difficili, cujus ſolutionem neque Euclides, neque Apollonius invenerant. Hanc vero Carteſius (a) novo Analyſeos præſidio poſt quinque, aut ſex hebdomadas extricavit, ibique Geometriæ ſuæ initia auſpicatus eſt, ubi veteres terminum ſibi ſtatuerant. Infinitos autem caſus quæſtio illa complectitur: ex quibus nos unum vel alterum ex facilioribus, quippe per circulum & parobolam conſtrui poſſint, expendemus, ne tyrones tam famoſæ quæſtionis notitia diutius lateat. En igitur qualis a Carteſio (b) luculenter exponitur, quæ a Pappo tantum fuerat verbis ſubobſcuris indicata.

Datis poſitione tribus, quatuorve, aut pluribus rectis lineis, quæritur primo punctum, a quo totidem aliæ rectæ lineæ ſingulæ ad ſingulas datarum duci poſſint, quæ cum ipſis datos efficiant angulos, & quarum rectangulum ſub duabus contentum datam habeat rationem ad quadratum tertiæ, ſi ſint tres; vel ad rectangulum reliquarum datarum, ſi ſint quatuor; aut ſi quinque ſint, ut parallelepipedum, quod ſub tribus ex illis comprehenditur, datam habeat rationem ad parallelepipedum, quod ſub duabus reliquis comprehenditur & alia quadam data. Aut ſi ſex ſint, ut parallelepipedum ſub tribus contentum datum habeat ratio-

(a) Epiſt. 71. Tom. 2. (b) Geometr. lib. 1. pag. m. 10.

rationem ad parallelepipedum ſub tribus reliquis comprehenſum &c. Atque ita porro quæſtionem hanc ad omnem alium linearum numerum extendere licet.

I. Datæ ſint (*Fig.* 46.) quinque rectæ inter ſe parallelæ *Aa*, *Bb*, *Dd*, *Ee*, *Ff*, quæritur punctum *C*, ex quo ſi ad quinque rectas datas ducantur *CA*, *CB*, *CD*, *CE*, *CF*, quæ cum prioribus faciant angulum datum gr. 80., parallelepipedum ſub $CA \times CD \times CE$ ſit ad parallelepipedum ſub $CB \times CF$ & recta data $2a$, ut 2 ad 1.

Eſto factum, & quia datis parallelis datur quoque earum diſtantia, ſeu rectæ *AB*, *BD*, *DE*, ſint hæ ſingulæ $= a$, $EF = \frac{1}{2} a$, & $CB = x$, erit $CA = x + a$, $CD = x - a$, $CE = x - 2a$, $CF = x - \frac{5}{2} a$. Atque hinc erit ex conditione problematis

$$x^3 - 2ax^2 - a^2 x + 2a^3 \cdot 2ax^2 - 5a^2 x :: 2 \cdot 1$$
$$x^3 - 2ax^2 - a^2 x + 2a^3 = 4ax^2 - 10a^2 x$$
$$x^3 - 6ax^2 + 9a^2 x + 2a^3 = 0$$

Hæc autem æquatio cum per nullum binomium dividi poſſit, tolli debet ſecundus terminus, ut per parabolam & circulum conſtrui poſſit. Fiat igitur $x - 2a = y$, erit $x = y + 2a$, factaque terminorum ſubſtitutione, oritur æquatio ſecundo termino carens *per Prop.* 5. *Cap.* 6.

$$y^3 * - 3a^2 y + 4a^3 = 0$$

Hinc ex formula generali $y^3 - px + q = 0$, erit $-3a^2 = -p$, $4a^3 = q$, deſcriptaque parabola cum parametro $a = 1$, tum etiam circulo omnino ut *in Propoſ.* 1. *hujus* factum eſt, invenietur *y*, cui ſi addantur $2a$, innoteſcet punctum $x \, (= y + 2a) = BC$. Ducta jam *Cc* ceteris *Aa*,

Bb &c.

Bb &c. parallela, ex quovis puncto ejusdem ducantur rectæ datas parallelas in angulo dato ſecantes, ſolvitur quæſtio.

II. Datæ ſint (*Fig. 47.*) poſitione ſeptem parallelæ, quarum ſit æqualis diſtantia $= a$, quæ faciant quemcunque angulum cum recta *AG*: quæritur punctum *M*, ita ut ſolidum $MC \times MA \times ME \times MG = MD \times MB \times MF \times 4a$. Sit $MD = x$, erit $MC = x + a$, $MB = x + 2a$, $MA = x + 3a$, $ME = MD - DE = x - a$, $MF = FE + ED - DM = 2a - x$, $MG = GD - DM = 3a - x$. Atque hinc oritur æquatio

$$-x^4 + 10a^2x^2 - 9a^4 = 16a^3x - 4ax^3$$
$$x^4 - 4ax^3 - 10a^2x^2 + 16a^3x + 9a^4 = 0$$

Quæ quidem cum per nullum binomium dividi poſſit, auferri debet ſecundus terminus, ut per parabolam & circulum conſtruatur. Itaque ſumatur $x - a = y$, erit $x = y + a$, & *per Prop. 5. Cap. 6.* oritur æquatio ſecundo termino carens.

$$y^4 * - 16a^2y^2 - 12a^3y + 12a^4 = 0$$

Et comparatis hujus terminis cum formula generali, erit $-16a^2 = -p$, $-12a^3 = -q$, $+12a^4 = r$. Proinde facile conſtruitur *per Prop. 3. hujus*, & invenitur y, cui ſi addatur $+ a$, innoteſcet punctum quæſitum $x = y + a$, a quo ducta *Mm* ceteris *Aa*, *Bb*, *Cc* &c. parallela, ex quocunque illius puncto ducatur recta parallelas *Aa*, *Bb*, *Cc* &c. ſecans in angulo dato, erit problema ſolutum.

FINIS.

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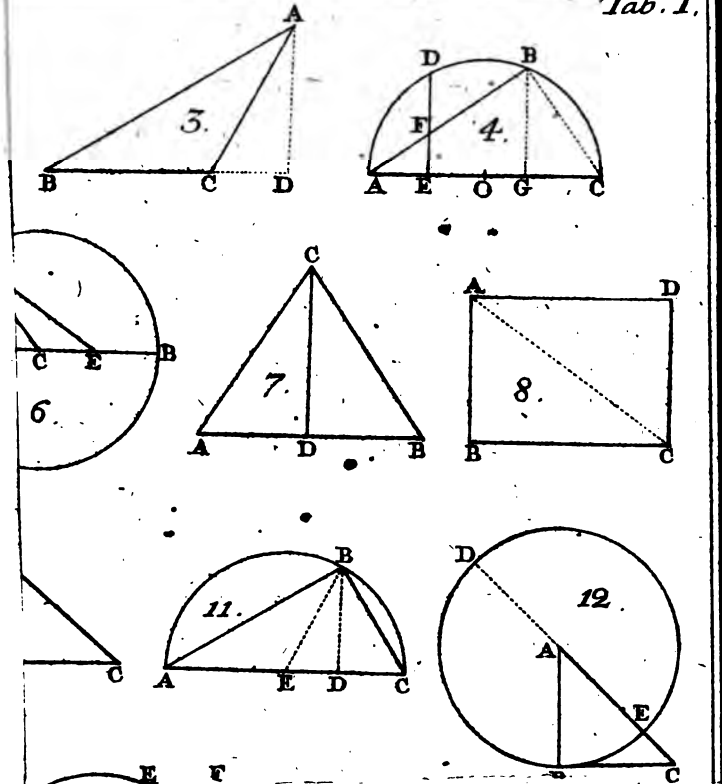
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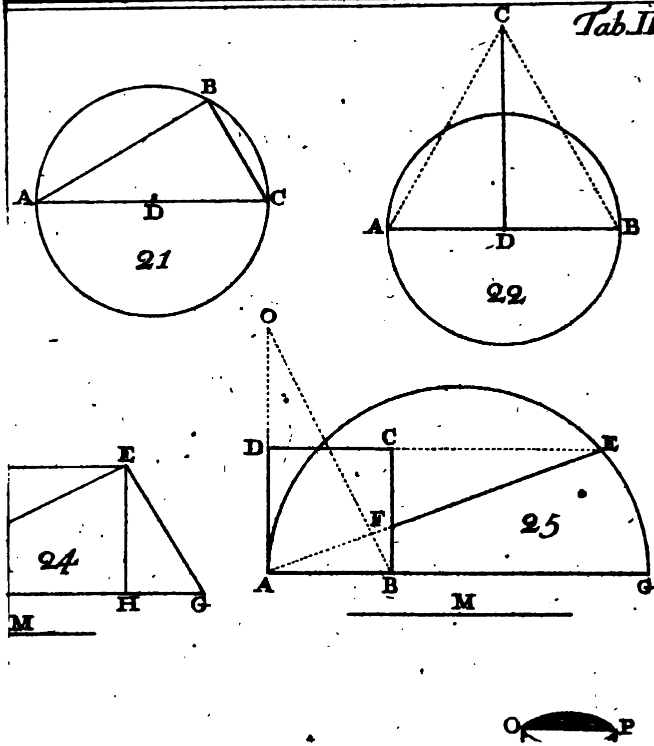
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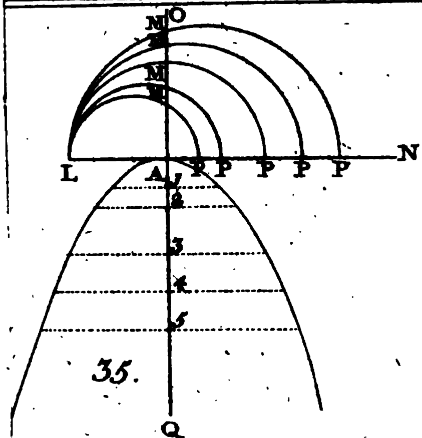
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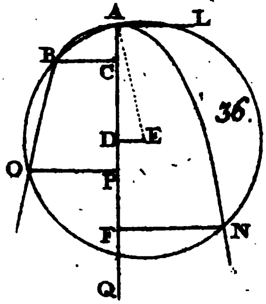


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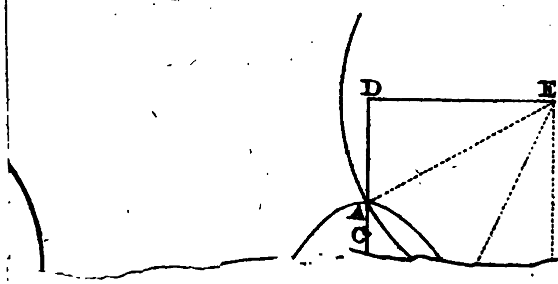




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Tab. IV.

